

# On the Cauchy problem of 2D viscous shallow water system in Besov spaces

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## Abstract

In this paper we consider the Cauchy problem for 2D viscous shallow water system in Besov spaces. We firstly prove the local well-posedness of this problem in  $B_{p,r}^s(\mathbb{R}^2)$ ,  $s > \frac{2}{p} + 1$  by using the Littlewood-Paley theory, the Bony decomposition and the theories of transport equations and transport diffusion equations. Then we give a blow-up criterion of solutions to the system in  $B_{p,r}^s$ ,  $s > \frac{2}{p} + 1$ . Moreover, by this blow-up criterion, we can prove the global existence of the system with small enough initial data in  $B_{p,r}^s(\mathbb{R}^2)$ ,  $p \leq 2$  and  $s > 1 + \frac{2}{p}$ . Our obtained results generalize and cover the recent results in [12].

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# 1 Introduction

We consider the following Cauchy problems for 2D viscous shallow water equations

$$(1.1) \quad \begin{cases} h(u_t + (u \cdot \nabla)u) - \nu \nabla \cdot (h \nabla u) + h \nabla h = 0, \\ h_t + \operatorname{div}(hu) = 0, \\ u|_{t=0} = u_0, \quad h|_{t=0} = h_0, \end{cases}$$

where  $h(x, t)$  is the height of fluid surface,  $u(x, t) = (u^1(x, t), u^2(x, t))$  is the horizontal velocity field,  $x = (x_1, x_2) \in \mathbb{R}^2$ , and  $0 < \nu < 1$  is the viscous coefficients. For the initial data  $h_0(x)$ , we suppose that it is a small perturbation of some positive constant  $\bar{h}_0$ . We study the Cauchy problems (1.1) in Besov space  $B_{p,r}^s(\mathbb{R}^2)$ ,  $s > \frac{2}{p} + 1$ . For the sake of convenience, we let the notation  $B_{p,r}^s$  stand for  $B_{p,r}^s(\mathbb{R}^2)$  in the following text, and also let the notations  $L^p$  and  $H^s$  stand for  $L^p(\mathbb{R}^2)$  and  $H^s(\mathbb{R}^2)$ , respectively.

Recently, Bresch et al. [3, 4] have systematically introduced the viscous shallow water equations. Bui in [5] proved the local existence and uniqueness of classical solutions to the Cauchy-Dirichlet problem for the shallow water equations with initial data in  $C^{2+\alpha}$  by using Lagrangian coordinates and Hölder space estimates. Kloeden in [8] and Sundbye in [10] independently showed the global existence and uniqueness of classical solutions to the Cauchy-Dirichlet problem using Sobolev space estimates by following the energy method of Matsumura and Nishida [9]. Sundbye in [11] proved the existence and uniqueness of classical solutions to the Cauchy problem using the method of [9].

Wang and Xu in [12] obtained local solutions for any initial data and global solutions for small initial data  $h_0 - \bar{h}_0, u_0 \in H^s$ ,  $s > 2$ . Haspot got global existence in time for small initial data  $h_0$ ,  $h_0 - \bar{h}_0 \in \dot{B}_{2,1}^0 \cap \dot{B}_{2,1}^1$  and  $u_0 \in \dot{B}_{2,1}^0$  as a special case in [7], and Chen, Miao and Zhang in [6] to prove the local existence in time for general initial data and the global existence in time for small initial data where  $h_0 - \bar{h}_0 \in \dot{B}_{2,1}^0 \cap \dot{B}_{2,1}^1$  and  $u_0 \in \dot{B}_{2,1}^0$  with additional conditions that  $h \geq h_0$  and  $h_0$  is a strictly positive constant.

In the paper, we mainly use the Littlewood-Paley theory, the Bony decomposition and the Besov space theories for transport equations and transport-diffusion equations to obtain the local existence and uniqueness of solutions for any initial data in  $B_{p,r}^s(\mathbb{R}^2)$ ,  $s > \frac{2}{p} + 1$ , and a blow-up criterion in  $B_{p,r}^s(\mathbb{R}^2)$ ,  $s > \frac{2}{p} + 1$ . Moreover, by the blow-up criterion, we can prove the global existence of the system with small enough initial data in  $B_{p,r}^s(\mathbb{R}^2)$ ,  $p \leq 2$  and  $s > 1 + \frac{2}{p}$ .

The main results of this paper are as follows:

**Theorem 1.1.** *Let  $u_0, h_0 - \bar{h}_0 \in B_{p,r}^s$ ,  $s > \frac{2}{p} + 1$ ,  $\|h_0 - \bar{h}_0\|_{B_{p,r}^s} < \bar{h}_0$ . Then there exists a positive time  $T$ , a unique solution  $(u, h)$  of the Cauchy problem (1.1) such that*

$$u, h - \bar{h}_0 \in \tilde{L}^\infty([0, T]; B_{p,r}^s) \cap C([0, T]; B_{p,r}^s), \quad u \in \tilde{L}^2([0, T]; B_{p,r}^{s+1}).$$

**Theorem 1.2.** *Let  $u_0, h_0 \in B_{p,r}^s \times B_{p,r}^s$ ,  $s > 1 + \frac{2}{p}$ , and let  $u, h$  be the corresponding solution of the Cauchy problem (1.1) in  $B_{p,r}^s \times B_{p,r}^s$ . Assume  $T^*$  is the maximal existence time of solution. If  $T^*$  is finite, then we have*

$$\int_0^{T^*} \|\nabla u\|_{L^\infty}^{r_1} + \|h\|_{L^\infty}^{r_1} + \|\nabla(\ln(1+h))\|_{L^\infty}^{r_1} dt' = \infty,$$

where  $r_1 = \max\{r', 2\}$ .

**Theorem 1.3.** *Let  $u_0, h_0 \in B_{p,r}^s$ ,  $p \leq 2$ ,  $s > 1 + \frac{2}{p}$ . If there exists an  $\varepsilon$  small enough such that  $\|u_0\|_{B_{p,r}^s} + \|h_0\|_{B_{p,r}^s} < \varepsilon$ , then the corresponding solution of the Cauchy problem (1.1) in  $B_{p,r}^s$  is global in time.*

## 2 Preliminaries

First of all, we transform the system (1.1). For a sake of convenience, we take  $\bar{h}_0 = 1$ . Substituting  $h$  by  $1 + h$  in (1.1), we have

$$(2.1) \quad \begin{cases} u_t + (u \cdot \nabla)u - \nu \Delta u - \nu \nabla(\ln(1+h)) \nabla u + \nabla h = 0, \\ h_t + \operatorname{div} u + \operatorname{div}(hu) = 0, \\ u|_{t=0} = u_0, \quad h|_{t=0} = h_0, \end{cases}$$

here  $h_0 \in B_{p,r}^s$ , and  $\|h_0\|_{B_{p,r}^s} \leq \frac{1}{8C_0C_{s,p}}$ ,  $C_0, C_{s,p}$  is determined below.

Then we introduce the Littlewood-Paley decomposition briefly.

**Proposition 2.1.** *[2] Littlewood-Paley Decomposition:*

Let  $\mathcal{B} = \{\xi \in \mathbb{R}^2, |\xi| \leq \frac{4}{3}\}$  be a ball, and  $\mathcal{C} = \{\xi \in \mathbb{R}^2, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$  be an annulus. There exist two radial functions  $\chi$  and  $\varphi$  valued in the interval  $[0, 1]$ , belonging respectively to  $\mathcal{D}(\mathcal{B})$  and  $\mathcal{D}(\mathcal{C})$ , such that

$$(2.2) \quad \forall \xi \in \mathbb{R}^2, \quad \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1,$$

$$(2.3) \quad \forall \xi \in \mathbb{R}^2 \setminus \{0\}, \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1,$$

$$(2.4) \quad |j - j'| \geq 2 \Rightarrow \operatorname{Supp} \varphi(2^j \cdot) \cap \operatorname{Supp} \varphi(2^{j'} \cdot) = \emptyset,$$

$$(2.5) \quad j \geq 2 \Rightarrow \operatorname{Supp} \chi \cap \operatorname{Supp} \varphi(2^j \cdot) = \emptyset,$$

the set  $\tilde{\mathcal{C}} \stackrel{\text{def}}{=} B(0, 2/3) + \mathcal{C}$  is an annulus, and we have

$$(2.6) \quad |j - j'| \geq 5 \Rightarrow 2^j \tilde{\mathcal{C}} \cap 2^{j'} \mathcal{C} = \emptyset.$$

Further, we have

$$(2.7) \quad \forall \xi \in \mathbb{R}^2, \quad \frac{1}{2} \leq \chi^2(\xi) + \sum_{j \geq 0} \varphi^2(2^{-j}\xi) \leq 1,$$

$$(2.8) \quad \forall \xi \in \mathbb{R}^2 \setminus \{0\}, \quad \frac{1}{2} \leq \sum_{j \in \mathbb{Z}} \varphi^2(2^{-j}\xi) \leq 1.$$

Now we can define the nonhomogeneous dyadic blocks  $\Delta_j$  and the nonhomogeneous low-frequency cut-off operator  $S_j$  as follows:

$$\begin{aligned}\Delta_j u &= 0, \text{ if } j \leq -2, \quad \Delta_{-1} u = \chi(D)u = \int_{\mathbb{R}^2} \tilde{h}(y)u(x-y)dy, \\ \Delta_j u &= \varphi(2^{-j}D)u = 2^{jd} \int_{\mathbb{R}^2} h(2^j y)u(x-y)dy, \quad \text{if } j \geq 0.\end{aligned}$$

and

$$S_j u = \sum_{j' \leq j-1} \Delta_{j'} u.$$

Where  $h = \mathcal{F}^{-1}\varphi$  and  $\tilde{h} = \mathcal{F}^{-1}\chi$ .

Next we define the Besov spaces:

**Definition 2.2.** [2] Let  $s \in \mathbb{R}$  and  $(p, r) \in [1, \infty]^2$ . The nonhomogeneous Besov space  $B_{p,r}^s$  consists of all tempered distribution  $u$  such that:

$$\left( \sum_{j \geq -1} (2^{js} \|\Delta_j u\|_{L^p}) \right)_{\ell^r} < \infty,$$

and naturally the Besov norm is defined as follows

$$\|u\|_{B_{p,r}^s} = \left( \sum_{j \geq -1} (2^{js} \|\Delta_j u\|_{L^p}) \right)_{\ell^r}.$$

**Definition 2.3.** [2] The Bony decomposition: The nonhomogeneous paraproduct of  $v$  by  $u$  is defined by

$$T_u v = \sum_j S_{j-1} u \Delta_j v.$$

The nonhomogeneous remainder of  $u$  by  $v$  is defined by

$$R(u, v) = \sum_{|k-j| \leq 1} \Delta_k u \Delta_j v.$$

The operators  $T$  and  $R$  are bilinear, and we have the following Bony decomposition

$$uv = T_v u + T_u v + R(u, v).$$

**Lemma 2.4.** [2] Bernstein-Type inequalities:

Let  $\mathcal{C}$  be an annulus and  $\mathcal{B}$  a ball. A constant  $C$  exists such that for any nonnegative integer  $k$ , any couple  $(p, q)$  in  $[1, \infty]^2$  with  $q \geq p \geq 1$ , and any function  $u$  of  $L^p$ , we have

$$\text{Supp } \hat{u} \subset \lambda \mathcal{B} \Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^q} \leq C^{k+1} \lambda^{k+d(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p},$$

$$\text{Supp } \hat{u} \subset \lambda \mathcal{C} \Rightarrow C^{-k-1} \lambda^k \|u\|_{L^p} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^q} \leq C^{k+1} \lambda^k \|u\|_{L^p}.$$

Then we give some properties of the Besov spaces which will be used in this paper.

**Lemma 2.5.** [2] *Let  $1 \leq p_1 \leq p_2 \leq \infty$  and  $1 \leq r_1 \leq r_2 \leq \infty$ . Then for any  $s \in \mathbb{R}$ , the space  $B_{p_1, r_1}^s$  is continuously embedded in  $B_{p_2, r_2}^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}$ . Obviously, we also have that the space  $B_{p, r}^{s_2}$  is continuously embedded in  $B_{p, r}^{s_1}$  and  $B_{p, \infty}^{s_2}$  is continuously embedded in  $B_{p, 1}^{s_1}$  if  $s_1 < s_2$ .*

**Lemma 2.6.** [2] *If  $u \in B_{p, r}^s$ , then  $\nabla u \in B_{p, r}^{s-1}$ , and we have*

$$\|\nabla u\|_{B_{p, r}^{s-1}} \leq C \|u\|_{B_{p, r}^s}.$$

**Lemma 2.7.** [2] *The set  $B_{p, r}^s$  is a Banach space and satisfies the Fatou property, namely, if  $(u_n)_{n \in \mathbb{N}}$  is a bounded sequence of  $B_{p, r}^s$ . Then an element  $u$  of  $B_{p, r}^s$  and a subsequence  $u_{\psi(n)}$  exist such that:*

$$\lim_{n \rightarrow \infty} u_{\psi(n)} = u \text{ in } \mathcal{S}', \quad \|u\|_{B_{p, r}^s} \leq C \liminf_{n \rightarrow \infty} \|u_{\psi(n)}\|_{B_{p, r}^s}.$$

**Lemma 2.8.** [2] *If  $s > \frac{d}{p}$  or  $s = \frac{d}{p}$ ,  $r = 1$ , then the  $B_{p, r}^s$  space is continuously embedded in  $L^\infty$ , i.e*

$$\|u\|_{L^\infty} \leq C_{s, p} \|u\|_{B_{p, r}^s}.$$

**Lemma 2.9.** [2] *Let  $f$  be a smooth function,  $f(0) = 0$ ,  $s > 0$ ,  $(p, r) \in [1, \infty]^2$ . If  $u \in B_{p, r}^s \cap L^\infty$ , then so does  $f \circ u$ , and we have*

$$\|f \circ u\|_{B_{p, r}^s} \leq C(s, f', \|u\|_{L^\infty}) \|u\|_{B_{p, r}^s}.$$

**Lemma 2.10.** [2] *A constant  $C$  exists which satisfies the following inequalities for any couple of real numbers  $(s, t)$  with  $t$  negative and any  $(p, r_1, r_2)$  in  $[1, \infty]^3$ :*

$$\begin{aligned} \|T\|_{\mathcal{L}(L^\infty \times B_{p, r}^s; B_{p, r}^s)} &\leq C^{|s|+1}, \\ \|T\|_{\mathcal{L}(B_{\infty, r_1}^t \times B_{p, r_2}^s; B_{p, r}^{s+t})} &\leq \frac{C^{|s+t|+1}}{-t} \quad \text{with} \quad \frac{1}{r} \stackrel{\text{def}}{=} \min\{1, \frac{1}{r_1} + \frac{1}{r_2}\}. \end{aligned}$$

**Lemma 2.11.** [2] *A constant  $C$  exists which satisfies the following inequalities. Let  $(s_1, s_2)$  be in  $\mathbb{R}^2$  and  $(p_1, p_2, r_1, r_2)$  be in  $[1, \infty]^4$ . Assume that*

$$\frac{1}{p} \stackrel{\text{def}}{=} \frac{1}{p_1} + \frac{1}{p_2} \leq 1 \quad \text{and} \quad \frac{1}{r} \stackrel{\text{def}}{=} \frac{1}{r_1} + \frac{1}{r_2} \leq 1.$$

*If  $s_1 + s_2 > 0$ , then we have, for any  $(u, v)$  in  $B_{p_1, r_1}^{s_1} \times B_{p_2, r_2}^{s_2}$ ,*

$$\|R(u, v)\|_{B_{p, r}^{s_1+s_2}} \leq \frac{C^{|s_1+s_2|+1}}{s_1+s_2} \|u\|_{B_{p_1, r_1}^{s_1}} \|v\|_{B_{p_2, r_2}^{s_2}}.$$

*If  $r = 1$  and  $s_1 + s_2 = 0$ , then we have, for any  $(u, v)$  in  $B_{p_1, r_1}^{s_1} \times B_{p_2, r_2}^{s_2}$ ,*

$$\|R(u, v)\|_{B_{p, \infty}^0} \leq C^{|s_1+s_2|+1} \|u\|_{B_{p_1, r_1}^{s_1}} \|v\|_{B_{p_2, r_2}^{s_2}}.$$

**Lemma 2.12.** [2] For any  $s > 0$  and  $(p, r) \in [1, \infty]^2$ , the space  $B_{p,r}^s \cap L^\infty$  is an algebra, and a constant exists such that:

$$\|uv\|_{B_{p,r}^s} \leq \frac{C^{s+1}}{s} \left( \|u\|_{L^\infty} \|v\|_{B_{p,r}^s} + \|v\|_{L^\infty} \|u\|_{B_{p,r}^s} \right).$$

Moreover, if  $s > \frac{d}{p}$  or  $s = \frac{d}{p}, r = 1$ , we have

$$\|uv\|_{B_{p,r}^s} \leq \frac{C^{s+1}}{s} \|u\|_{B_{p,r}^s} \|v\|_{B_{p,r}^s}.$$

For the transport equations

$$(2.9) \quad \begin{cases} \partial_t f + v \cdot \nabla f = g \\ f|_{t=0} = f_0, \end{cases}$$

we have

**Lemma 2.13.** [2] Let  $1 \leq p \leq p_1 \leq \infty$ ,  $1 \leq r \leq \infty$ . Assume that

$$(2.10) \quad s \geq -d \min\left(\frac{1}{p_1}, \frac{1}{p'}\right) \quad \text{or} \quad s \geq -1 - d \min\left(\frac{1}{p_1}, \frac{1}{p'}\right) \quad \text{if } \operatorname{div} v = 0$$

with strict inequality if  $r < \infty$ . There exists a constant  $C$ , depending only on  $d, p, p_1, r$  and  $s$ , such that for all solutions  $f \in L^\infty([0, T]; B_{p,r}^s)$  of (2.9), initial data  $f_0$  in  $B_{p,r}^s$ , and  $g$  in  $L^1([0, T]; B_{p,r}^s)$ , we have, for a.e.  $t \in [0, T]$ ,

$$\|f\|_{\tilde{L}_t^\infty(B_{p,r}^s)} \leq \left( \|f_0\|_{B_{p,r}^s} + \int_0^t \exp(-CV_{p_1}(t')) \|g(t')\|_{B_{p,r}^s} dt' \right) \exp(CV_{p_1}(t))$$

with, if the inequality is strict in (2.10),

$$(2.11) \quad V'_{p_1}(t) = \begin{cases} \|\nabla v(t)\|_{B_{p_1,r}^{s-1}}, & \text{if } s > 1 + \frac{d}{p_1} \text{ or } s = 1 + \frac{d}{p_1}, r = 1, \\ \|\nabla v(t)\|_{B_{p_1,\infty}^{\frac{d}{p_1}} \cap L^\infty}, & \text{if } s < 1 + \frac{d}{p_1} \end{cases}$$

and, if equality holds in (2.10) and  $r = \infty$ ,

$$V'_{p_1} = \|\nabla v(t)\|_{B_{p_1,1}^{\frac{d}{p_1}}}.$$

If  $f = v$ , then for all  $s > 0$  ( $s > -1$ , if  $\operatorname{div} u = 0$ ), the estimate holds with

$$V'_{p_1}(t) = \|\nabla u\|_{L^\infty},$$

where  $\|u\|_{\tilde{L}_T^p(B_{p,r}^s)}$  is defined in Lemma 2.15.

For the transport diffusion equations

$$(2.12) \quad \begin{cases} \partial_t f + v \cdot \nabla f - \nu \Delta f = g \\ f|_{t=0} = f_0, \end{cases}$$

we have the following lemma.

**Lemma 2.14.** [2] *Let  $1 \leq p_1 \leq p \leq \infty$ ,  $1 \leq r \leq \infty$ ,  $s \in \mathbb{R}$  satisfy (2.10), and  $V_{p_1}$  be defined as in Lemma 2.8.*

*There exists a constant  $C$  which depends only on  $d, r, s$  and  $s - 1 - \frac{d}{p_1}$  and is such that for any smooth solution of (11) and  $1 \leq \rho_1 \leq \rho \leq \infty$ , we have*

$$\begin{aligned} \nu^{\frac{1}{\rho}} \|f\|_{\tilde{L}_T^\rho(B_{p,r}^{s+\frac{2}{\rho}})} &\leq C e^{C(1+\nu T)^{\frac{1}{\rho}} V_{p_1}(T)} \left( (1+\nu T)^{\frac{1}{\rho}} \|f_0\|_{B_{p,r}^s} \right. \\ &\quad \left. + (1+\nu T)^{1+\frac{1}{\rho}-\frac{1}{\rho_1}} \nu^{\frac{1}{\rho_1}-1} \|g\|_{\tilde{L}_T^{\rho_1}(B_{p,r}^{s-2+\frac{2}{\rho_1}})} \right). \end{aligned}$$

For the space  $\tilde{L}_T^\rho(B_{p,r}^s)$ , we have the following properties:

**Lemma 2.15.** [2] *For all  $T > 0$ ,  $s \in \mathbb{R}$ , and  $1 \leq r, \rho \leq \infty$ , we set*

$$\|u\|_{\tilde{L}_T^\rho(B_{p,r}^s)} \stackrel{def}{=} \|2^{js} \|\Delta_j u\|_{L_T^\rho(L^p)}\|_{l^r(\mathbb{Z})}.$$

*We can then define the space  $\tilde{L}_T^\rho(B_{p,r}^s)$  as the set of tempered distributions  $u$  over  $(0, T) \times \mathbb{R}^d$  such that  $\|u\|_{\tilde{L}_T^\rho(B_{p,r}^s)} \leq \infty$ . By the Minkowski inequality, we have*

$$\|u\|_{\tilde{L}_T^\rho(B_{p,r}^s)} \leq \|u\|_{L_T^\rho(B_{p,r}^s)} \text{ if } r \geq \rho$$

$$\|u\|_{L_T^\rho(B_{p,r}^s)} \leq \|u\|_{\tilde{L}_T^\rho(B_{p,r}^s)} \text{ if } r \leq \rho.$$

*The general principle is that all the properties of continuity for the product, composition, remainder, and paraproduct remain true in those space.*

*Moreover when  $s > 0$ ,  $1 \leq p \leq \infty$ ,  $1 \leq \rho, \rho_1, \rho_2, \rho_3, \rho_4 \leq \infty$ , and*

$$\frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{1}{\rho_3} + \frac{1}{\rho_4},$$

*we have*

$$\|uv\|_{\tilde{L}_T^\rho(B_{p,r}^s)} \leq C \left( \|u\|_{\tilde{L}_T^{\rho_1}(L^\infty)} \|v\|_{\tilde{L}_T^{\rho_2}(B_{p,r}^s)} + \|v\|_{\tilde{L}_T^{\rho_3}(L^\infty)} \|u\|_{\tilde{L}_T^{\rho_4}(B_{p,r}^s)} \right).$$

**Lemma 2.16.** [2] *The time-space estimate for heat equation:*

*Let  $\mathcal{C}$  be an annulus and  $\lambda$  a positive real number. Let  $u_0, f$  satisfy  $\text{Supp } \hat{u}_0, \text{Supp } \hat{f}(t) \subset \lambda \mathcal{C}$  for all  $t \in [0, T]$ . Consider  $u$ , a solution of*

$$\partial_t u - \nu \Delta u = f \quad \text{and} \quad u|_{t=0} = u_0.$$



Then there exists a positive constant  $C$ , depending only on  $\mathcal{C}$ , such that for any  $1 \leq a \leq b \leq \infty$  and  $1 \leq p \leq q \leq \infty$ , we have

$$\|u\|_{L_T^q(L^b)} \leq C(\nu\lambda^2)^{-\frac{1}{q}}\lambda^{d(\frac{1}{a}-\frac{1}{b})}\|u_0\|_{L^a} + C(\nu\lambda^2)^{-1+(\frac{1}{p}-\frac{1}{q})}\lambda^{d(\frac{1}{a}-\frac{1}{b})}\|f\|_{L_T^p(L^a)}.$$

**Lemma 2.17.** [2] *The commutator estimate:*

Let  $s \in \mathbb{R}$ ,  $1 \leq r \leq \infty$ , and  $1 \leq p \leq p_1 \leq \infty$ . Let  $v$  be a vector field over  $\mathbb{R}^d$ . Assume that

$$s > -d \min\left\{\frac{1}{p_1}, \frac{1}{p'}\right\} \quad \text{or} \quad s > -1 - d \min\left\{\frac{1}{p_1}, \frac{1}{p'}\right\} \quad \text{if} \quad \operatorname{div} v = 0.$$

Define  $R_j(v, f) \stackrel{\text{def}}{=} [v \cdot \nabla, \Delta_j]f$  or  $R_j(v, f) \stackrel{\text{def}}{=} \operatorname{div}([v, \Delta_j]f)$ , if  $\operatorname{div} v = 0$ . There exists a constant  $C$ , depending continuously on  $p, p_1, s$  and  $d$ , such that

$$\|(2^{js}\|R_j\|_{L^p})_j\|_{\ell^r} \leq C\|\nabla v\|_{B_{p_1, \infty}^p \cap L^\infty} \|f\|_{B_{p, r}^s} \quad \text{if} \quad s < 1 + \frac{d}{p_1}.$$

Further, if  $s > 0$  (or  $s > -1$  if  $\operatorname{div} v = 0$ ) and  $\frac{1}{p_2} = \frac{1}{p} - \frac{1}{p_1}$ , then

$$\|(2^{js}\|R_j\|_{L^p})_j\|_{\ell^r} \leq C \left( \|\nabla v\|_{L^\infty} \|f\|_{B_{p, r}^s} + \|\nabla f\|_{L^{p_2}} \|\nabla v\|_{B_{p_1, r}^{s-1}} \right).$$

**Lemma 2.18.** [12] *Let  $s > 2$ ,  $u_0, h_0 \in H^s$ . Then there exist a positive time  $T$ , a unique solution  $(u, h)$  of the Cauchy problem (2.1) such that*

$$u, h \in L^\infty([0, T], H^s), \nabla u \in L^2([0, T]; H^s).$$

Furthermore, there exists a constant  $c$  such that if  $\|u_0\|_{H^s} + \|h_0\|_{H^s} \leq c$ , then  $T = \infty$ .

**Remark 2.19.** For the sake of convenient, for the fixed  $s, p, r$ , we let  $C_0(> 1)$  be the maximum constant of Lemmas 2.6-2.18.

### 3 The local existence

In order to study the local existence of solution, we define the function set  $(u, h) \in \chi([0, T], s, p, r, E_1, E_2)$ , if  $(u, h) \in \tilde{L}^\infty([0, T]; B_{p, r}^s)$ , and

$$\|u\|_{\tilde{L}^\infty([0, T]; B_{p, r}^s)} \leq E_1, \quad \|h\|_{\tilde{L}^\infty([0, T]; B_{p, r}^s)} \leq E_2,$$

where

$$E_1 = 4C_0\|u_0\|_{B_{p, r}^s}, \quad E_2 = 4C_0\|h_0\|_{B_{p, r}^s}.$$

Next, we will prove Theorem 1.1 by the method of successive approximations. Let us define the sequence  $(u_n, h_n)$  by the following linear system:

$$(3.1) \quad \begin{cases} (u_1, h_1) = S_2(u_0, h_0), \\ \partial_t u_{n+1} + (u_n \cdot \nabla) u_{n+1} - \nu \Delta u_n = \frac{\nu}{1+h_n} \nabla h_n \nabla u_n + \nabla h_n, \\ \partial_t h_{n+1} + (u_n \cdot \nabla) h_{n+1} = -\operatorname{div} u_n - h \operatorname{div} u_n, \\ (u_{n+1}, h_{n+1})|_{t=0} = S_{n+2}(u_0, h_0). \end{cases}$$

Since  $S_q$  are smooth operators, the initial data  $S_{n+2}(u_0, h_0)$  are smooth functions. If  $(u_n, h_n) \in \chi([0, T], s, p, r, E_1, E_2)$  are smooth, then we have that for any  $t \in [0, T]$ ,

$$\|h_n\|_{L^\infty} \leq C_{s,p} \|h_n\|_{B_{p,r}^s} \leq C_{s,p} E_2 = 4C_0 C_{s,p} \|h_0\|_{B_{p,r}^s} \leq \frac{4C_0 C_{s,p}}{8C_0 C_{s,p}} = \frac{1}{2}.$$

Thus  $\frac{\nu}{1+h_n} \nabla h_n \nabla u_n + \nabla h_n$  and  $-\operatorname{div} u_n - h \operatorname{div} u_n$  are also smooth functions. Note that the first equation in (3.1) is a transport diffusion equation for  $u_{n+1}$ , and the second equation is a transport equation for  $h_{n+1}$ . Then the local existence of the smooth function for the Cauchy problem (3.1) is obvious.

We split the proof of Theorem 1.1 into two steps: Estimation for big norms and Convergence for small norms.

### 3.1. Estimation for big norms

In this subsection, we want to prove the following proposition.

**Proposition 3.1.** *Suppose that  $(u_0, h_0) \in B_{p,r}^s \times B_{p,r}^s$ ,  $s > 1 + \frac{2}{p}$  and  $\|h_0\|_{B_{p,r}^s} \leq \frac{1}{8C_{s,p}}$ , then there exists a positive time  $T_1$ , such that for any  $n \in \mathbb{N}$ ,  $(u_n, h_n) \in \chi([0, T_1], s, p, r, E_1, E_2)$ .*

Proof: Let  $T(\geq T_1)$  satisfy

$$T \leq 1, \quad e^{C_0^2 E_1 T} \leq 2, \quad e^{2C_0 E_1 T} \leq 2, \quad (1 + \nu T)^{\frac{3}{2}} \leq 2.$$

Then we prove the proposition by induction. Firstly let  $(u_1, h_1) = S_2(u_0, h_0)$ . Thus we have

$$\|u_1\|_{\tilde{L}_{T_1}^\infty(B_{p,r}^s)} \leq \|u_0\|_{B_{p,r}^s} \leq E_1, \quad \|h_1\|_{\tilde{L}_{T_1}^\infty(B_{p,r}^s)} \leq \|h_0\|_{B_{p,r}^s} \leq E_2.$$

If

$$\|u_n\|_{\tilde{L}_{T_1}^\infty(B_{p,r}^s)} \leq E_1, \quad \|h_n\|_{\tilde{L}_{T_1}^\infty(B_{p,r}^s)} \leq E_2,$$

then for  $h_{n+1}$ , in the view of Lemma 2.8, Lemma 2.13 and Lemma 2.15, for all  $t < T_1$ , we have

$$\begin{aligned}
(3.2) \quad & \|h_{n+1}\|_{\tilde{L}_t^\infty(B_{p,r}^s)} \leq \left( \|S_{n+2}h_0\|_{B_{p,r}^s} + \|\operatorname{div} u_n\|_{\tilde{L}_t^1(B_{p,r}^s)} + \|h_n \operatorname{div} u_n\|_{\tilde{L}_t^1(B_{p,r}^s)} \right) \\
& \exp\left(C_0 \int_0^t \|\nabla u_n(t')\|_{B_{p,r}^{s-1}} dt'\right) \\
& \leq 2\left(\frac{E_2}{4} + t^{\frac{1}{2}} \|\operatorname{div} u_n\|_{\tilde{L}_t^2(B_{p,r}^s)} + \|h_n\|_{\tilde{L}_t^2(B_{p,r}^s)} \|\operatorname{div} u_n\|_{L_t^2(L^\infty)} \right. \\
& \quad \left. + \|\operatorname{div} u_n\|_{\tilde{L}_t^2(B_{p,r}^s)} \|h_n\|_{L_t^2(L^\infty)} \right) \\
& \leq 2\left(\frac{E_2}{4} + t^{\frac{1}{2}} \|\operatorname{div} u_n\|_{\tilde{L}_t^2(B_{p,r}^s)} + Ct \|h_n\|_{\tilde{L}_t^\infty(B_{p,r}^s)} \|u_n\|_{\tilde{L}_t^\infty(B_{p,r}^s)} \right. \\
& \quad \left. + Ct^{\frac{1}{2}} \|\operatorname{div} u_n\|_{\tilde{L}_t^2(B_{p,r}^s)} \|h_n\|_{L_t^\infty(B_{p,r}^s)} \right) \\
& \leq \frac{E_2}{2} + CtE_1E_2 + C(1 + E_2)t^{\frac{1}{2}} \|u_n\|_{\tilde{L}_t^2(B_{p,r}^{s+1})}.
\end{aligned}$$

Now, we estimate  $\|u_n\|_{\tilde{L}_t^2(B_{p,r}^{s+1})}$ . By Lemmas 2.8-2.9, and Lemmas 2.14-2.15, we get

$$\begin{aligned}
(3.3) \quad & \|u_n\|_{\tilde{L}_t^2(B_{p,r}^{s+1})} \leq \nu^{-\frac{1}{2}} C_0 e^{(1+\nu t)^{\frac{1}{2}} \int_0^t \|\nabla u_{n-1}\|_{B_{p,r}^{s-1}} dt'} \left( (1 + \nu t)^{\frac{1}{2}} \|S_{n+1}u_0\|_{B_{p,r}^s} \right. \\
& \quad \left. + \nu^{-1}(1 + \nu t)^{\frac{3}{2}} \left( \|\nu \frac{\nabla h_{n-1} \nabla u_{n-1}}{1+h_{n-1}}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-2})} + \|\nabla h_{n-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-2})} \right) \right) \\
& \leq C \left( \|u_0\|_{B_{p,r}^s} + E_2 + \|\nabla(\ln(1 + h_{n-1}))\nabla u_{n-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \right) \\
& \leq C \left( \|u_0\|_{B_{p,r}^s} + E_2 + (\|\nabla u_{n-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \|\nabla(\ln(1 + h_{n-1}))\|_{\tilde{L}_t^\infty(L^\infty)} + \right. \\
& \quad \left. \|\nabla(\ln(1 + h_{n-1}))\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \|\nabla u_{n-1}\|_{\tilde{L}_t^\infty(L^\infty)} \right) \\
& \leq C(E_1 + E_2 + E_1E_2).
\end{aligned}$$

Thus letting

$T'_1 = \min\{T, (4CE_1)^{-1}, (4C^2(1 + E_2)(E_1 + E_2 + E_1E_2))^{-2}E_2^2\}$ , we can get that, for any  $t \leq T_1 \leq T'_1$ ,

$$\|h_{n+1}\|_{\tilde{L}_t^\infty(B_{p,r}^s)} \leq E_1.$$

For  $u_{n+1}$ , from Lemmas 2.8-2.9, and Lemmas 2.14-2.15, we obtain

(3.4)

$$\begin{aligned}
\|u_{n+1}\|_{\tilde{L}_t^\infty(B_{p,r}^s)} &\leq C_0 e^{C_0 \int_0^t \|\nabla u_n\|_{B_{p,r}^{s-1}} dt'} \times \\
&\left( \|u_0\|_{B_{p,r}^s} + (1 + \nu t)^{\frac{1}{2}} \nu^{-\frac{1}{2}} \left\| \nu \frac{\nabla h_n \nabla u_n}{1+h_n} + \nabla h_n \right\|_{\tilde{L}_t^2(B_{p,r}^{s-1})} \right) \\
&\leq 2C_0 \left( \|u_0\|_{B_{p,r}^s} + \left( \frac{1+\nu t}{\nu} \right)^{\frac{1}{2}} t^{\frac{1}{2}} \left( \|\nabla h_n\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} + \|\nu \nabla(\ln(1+h_n)) \nabla u_n\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \right) \right) \\
&\leq 2C_0 \left\{ \|u_0\|_{B_{p,r}^s} + \left( \frac{1+\nu t}{\nu} \right)^{\frac{1}{2}} t^{\frac{1}{2}} \left( \|\nabla h_n\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} + \nu C (\|\nabla u_n\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \|\nabla(\ln(1+h_n))\|_{\tilde{L}_t^\infty(L^\infty)} \right. \right. \\
&\quad \left. \left. + \|\nabla(\ln(1+h_n))\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \|\nabla u_n\|_{\tilde{L}_t^\infty(L^\infty)} \right) \right\} \\
&\leq 2C_0 (\|\nabla u_0\|_{B_{p,r}^s} + C t^{\frac{1}{2}} (E_2 + E_1 E_2)) \\
&\leq \frac{E_1}{2} + C t^{\frac{1}{2}} (E_2 + E_1 E_2).
\end{aligned}$$

Then, let

$$T_1 = \min\{T_1', (2C(E_2 + E_1 E_2))^{-2} E_1^2\}.$$

Thus, for all  $t \leq T_1$ , we have

$$\|u_{n+1}\|_{\tilde{L}_t^\infty(B_{p,r}^s)} \leq E_1.$$

This completes the proof of Proposition 3.1.

**Remark 3.2.** From the proof of Proposition 3.1, we can see that  $\|u_{n+1}\|_{\tilde{L}_{T_1}^2(B_{p,r}^{s+1})}$  is bounded by  $E_1$  uniformly.

### 3.2. Convergence of small norms

**Proposition 3.3.** Suppose that  $(u_0, h_0) \in B_{p,r}^s \times B_{p,r}^s$ ,  $s > 1 + \frac{2}{p}$  and  $\|h_0\|_{B_{p,r}^s} \leq \frac{1}{8CC_{s,p}}$ , then there exists a positive time  $T_2(\leq T_1)$ , such that  $(u_n, h_n)$  is a Cauchy sequence in  $\chi([0, T_2], s-1, p, r, E_1, E_2)$ .

Proof: From the equations in (3.1), we have

$$(3.5) \quad \begin{cases} \partial_t(u_{n+1} - u_n) + (u_n \cdot \nabla)(u_{n+1} - u_n) - \nu \Delta(u_{n+1} - u_n) = \sum_{j=1}^5 F_j \\ \partial_t((h_{n+1} - h_n) + (u_n \cdot \nabla)(h_{n+1} - h_n)) = \sum_{j=1}^4 J_j, \\ (u_{n+1} - u_n, h_{n+1} - h_n)|_{t=0} = \Delta_{n+1}(u_0, h_0), \end{cases}$$

where

$$\begin{aligned}
\sum_{j=1}^5 F_j &= (u_n - u_{n-1}) \cdot \nabla u_n + \nabla(h_n - h_{n-1}) + \frac{\nu}{1+h_n} \nabla h_n \nabla(u_n - u_{n-1}) \\
&+ \frac{\nu}{1+h_n} \nabla u_{n-1} \nabla(h_n - h_{n-1}) + \nu \left( \frac{1}{1+h_n} - \frac{1}{1+h_{n-1}} \right) \nabla h_{n-1} \nabla u_{n-1}, \\
\sum_{j=1}^4 J_j &= (u_n - u_{n-1}) \cdot \nabla h_n + \operatorname{div}(u_n - u_{n-1}) + h_n \operatorname{div}(u_n - u_{n-1}) \\
&+ (h_n - h_{n-1}) \operatorname{div} u_{n-1}.
\end{aligned} \tag{3.6}$$

Then we estimate the Besov norm of  $u_{n+1} - u_n$  and  $h_{n+1} - h_n$ . For any  $t \leq T_2 \leq T_1$ , by Lemma 2.14, we have

$$\begin{aligned}
\|u_{n+1} - u_n\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} &\leq C_0 e^{\int_0^t \|\nabla u_n\|_{B_{p,r}^{s-1}} dt'} \times \\
&\left( \|S_{n+2}u_0 - S_{n+1}u_0\|_{B_{p,r}^{s-1}} + \left(\frac{1+\nu t}{\nu}\right)^{\frac{1}{2}} \left\| \sum_{j=1}^5 F_j \right\|_{\tilde{L}_t^2(B_{p,r}^{s-1})} \right) \\
&\leq 2C_0 \left( \|\Delta_{n+1}u_0\|_{B_{p,r}^{s-1}} + \left\| \sum_{j=1}^5 F_j \right\|_{\tilde{L}_t^2(B_{p,r}^{s-2})} \right),
\end{aligned} \tag{3.7}$$

here

$$\begin{aligned}
\left\| \sum_{j=1}^5 F_j \right\|_{\tilde{L}_t^2(B_{p,r}^{s-2})} &\leq \|(u_n - u_{n-1}) \cdot \nabla u_n\|_{\tilde{L}_t^2(B_{p,r}^{s-2})} \\
&+ \|\nabla(h_n - h_{n-1})\|_{\tilde{L}_t^2(B_{p,r}^{s-2})} + \left\| \nu \frac{\nabla h_n}{1+h_n} \nabla(u_n - u_{n-1}) \right\|_{\tilde{L}_t^2(B_{p,r}^{s-2})} \\
&+ \left\| \nu \frac{\nabla(h_n - h_{n-1})}{1+h_n} \nabla u_{n-1} \right\|_{\tilde{L}_t^2(B_{p,r}^{s-2})} + \left\| \nu \frac{h_n - h_{n-1}}{(1+h_n)(1+h_{n-1})} \nabla h_{n-1} \nabla u_{n-1} \right\|_{\tilde{L}_t^2(B_{p,r}^{s-2})} \\
&= I_1 + I_2 + I_3 + I_4 + I_5,
\end{aligned} \tag{3.8}$$

Next, we deal with  $I_j$ ,  $j = 1, 2, 3, 4, 5$  term by term.

By Lemmas 2.5-2.6, Lemma 2.8 and Lemma 2.15, we have

$$\begin{aligned}
I_1 &\leq t^{\frac{1}{2}} \|(u_n - u_{n-1}) \cdot \nabla u_n\|_{\tilde{L}_t^\infty(B_{p,r}^{s-2})} \\
&\leq C t^{\frac{1}{2}} \|(u_n - u_{n-1}) \cdot \nabla u_n\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \\
&\leq C t^{\frac{1}{2}} (\|u_n - u_{n-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \|\nabla u_n\|_{L_t^\infty(L^\infty)} + \|\nabla u_n\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \|u_n - u_{n-1}\|_{L_t^\infty(L^\infty)}) \\
&\leq C t^{\frac{1}{2}} \|u_n - u_{n-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \|u_n\|_{\tilde{L}_t^\infty(B_{p,r}^s)} \\
&\leq C E_2 t^{\frac{1}{2}} \|u_n - u_{n-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})}.
\end{aligned} \tag{3.9}$$

From Lemma 2.6, it's easy to see that

$$I_2 \leq t^{\frac{1}{2}} \|\nabla(h_n - h_{n-1})\|_{\tilde{L}_t^\infty(B_{p,r}^{s-2})} \leq C t^{\frac{1}{2}} \|h_n - h_{n-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})}. \tag{3.10}$$

In view of Lemmas 2.5-2.6, Lemmas 2.8-2.9 and Lemma 2.15, we get

$$\begin{aligned}
(3.11) \quad I_3 &= \left\| \nu \frac{\nabla h_n}{1+h_n} \nabla(u_n - u_{n-1}) \right\|_{\tilde{L}_t^2(B_{p,r}^{s-2})} \\
&= \left\| \nu \nabla(\ln(1+h_n)) \nabla(u_n - u_{n-1}) \right\|_{\tilde{L}_t^\infty(B_{p,r}^{s-2})} \\
&\leq \left\| \nu T_{\nabla(\ln(1+h_n))} \nabla(u_n - u_{n-1}) \right\|_{\tilde{L}_t^\infty(B_{p,r}^{s-2})} + \left\| \nu T_{\nabla(u_n - u_{n-1})} \nabla(\ln(1+h_n)) \right\|_{\tilde{L}_t^\infty(B_{p,r}^{s-2})} \\
&\quad + \left\| \nu R(\nabla(\ln(1+h_n)), \nabla(u_n - u_{n-1})) \right\|_{\tilde{L}_t^2(B_{p,r}^{s-2})} \\
&= I_{31} + I_{32} + I_{33}.
\end{aligned}$$

By Lemma 2.10, we have

$$\begin{aligned}
(3.12) \quad I_{31} &\leq C t^{\frac{1}{2}} \left\| \nabla(\ln(1+h)) \right\|_{L_t^\infty(L^\infty)} \left\| \nabla(u_n - u_{n-1}) \right\|_{\tilde{L}_t^\infty(B_{p,r}^{s-2})} \\
&\leq C E_2 t^{\frac{1}{2}} \left\| u_n - u_{n-1} \right\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})}.
\end{aligned}$$

Still by Lemma 2.10, when  $s \leq 2 + \frac{2}{p}$ , we have

$$\begin{aligned}
(3.13) \quad I_{32} &\leq C t^{\frac{1}{2}} \left\| \nabla(u_n - u_{n-1}) \right\|_{\tilde{L}_t^\infty(B_{\infty,r}^{s-2-\frac{2}{p}-\varepsilon})} \left\| \nabla(\ln(1+h)) \right\|_{\tilde{L}_t^\infty(B_{p,\infty}^{\frac{2}{p}+\varepsilon})} \\
&\leq C E_2 t^{\frac{1}{2}} \left\| u_n - u_{n-1} \right\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})},
\end{aligned}$$

where  $\varepsilon$  is a positive real number and small enough, and it equals 0 when  $s = 1 + \frac{d}{p}$ . When  $s > 2 + \frac{2}{p}$ , we also have

$$\begin{aligned}
(3.14) \quad I_{32} &\leq C t^{\frac{1}{2}} \left\| \nabla(u_n - u_{n-1}) \right\|_{\tilde{L}_t^\infty(L^\infty)} \left\| \nabla(\ln(1+h)) \right\|_{\tilde{L}_t^\infty(B_{p,r}^{s-2})} \\
&\leq C E_2 t^{\frac{1}{2}} \left\| u_n - u_{n-1} \right\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})}.
\end{aligned}$$

For  $I_{33}$ , when  $s - 2 + \frac{2}{p} > 0$ , from Lemma 2.11, we have

$$\begin{aligned}
(3.15) \quad I_{33} &\leq C t^{\frac{1}{2}} \left\| R(\nabla(u_n - u_{n-1}), \nabla(\ln(1+h_n))) \right\|_{\tilde{L}_t^\infty(B_{p,r}^{s-2})} \\
&\leq C t^{\frac{1}{2}} \left\| R(\nabla(u_n - u_{n-1}), \nabla(\ln(1+h_n))) \right\|_{\tilde{L}_t^\infty(B_{\frac{p}{2},r}^{s-2+\frac{2}{p}})} \\
&\leq C t^{\frac{1}{2}} \left\| \nabla(u_n - u_{n-1}) \right\|_{\tilde{L}_t^\infty(B_{p,r}^{s-2})} \left\| \nabla(\ln(1+h_n)) \right\|_{\tilde{L}_t^\infty(B_{p,r}^{\frac{2}{p}})} \\
&\leq C t^{\frac{1}{2}} \left\| u_n - u_{n-1} \right\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \left\| \ln(1+h_n) \right\|_{\tilde{L}_t^\infty(B_{p,r}^{1+\frac{2}{p}})} \\
&\leq C E_2 t^{\frac{1}{2}} \left\| u_n - u_{n-1} \right\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})},
\end{aligned}$$

otherwise, we have  $1 < s \leq 2$ , then we get

$$\begin{aligned}
(3.16) \quad I_{33} &\leq C \|R(\nabla(u_n - u_{n-1}), \nabla(\ln(1 + h_n)))\|_{\tilde{L}_t^\infty(B_{\frac{p}{2}, r}^{s-2+\frac{2}{p}})} \\
&\leq C \|R(\nabla(u_n - u_{n-1}), \nabla(\ln(1 + h_n)))\|_{\tilde{L}_t^\infty(B_{\frac{p}{2}, r}^\varepsilon)} \\
&\leq C \|\nabla(u_n - u_{n-1})\|_{\tilde{L}_t^2(B_{p,r}^{1-s+\varepsilon})} \|\nabla(\ln(1 + h_n))\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \\
&\leq C t^{\frac{s-1}{2}} \|h_n\|_{\tilde{L}_t^\infty(B_{p,r}^s)} \|u_n - u_{n-1}\|_{\tilde{L}_t^{\frac{2}{2-s}}(B_{p,r}^1)} \\
&\leq C E_2 t^{\frac{s-1}{2}} \|u_n - u_{n-1}\|_{\tilde{L}^2(B_{p,r}^s)}^{2-s} \|u_n - u_{n-1}\|_{\tilde{L}^\infty(B_{p,r}^{s-1})}^{s-1}.
\end{aligned}$$

Then we deal with  $I_4$  by the similar method.

$$\begin{aligned}
(3.17) \quad I_4 &= \|\nu \frac{\nabla(h_n - h_{n-1})}{1+h_n} \nabla u_{n-1}\|_{\tilde{L}_t^2(B_{p,r}^{s-2})} \\
&= \nu \|(1 - \frac{h_n}{1+h_n}) \nabla(h_n - h_{n-1}) \nabla u_{n-1}\|_{\tilde{L}_t^2(B_{p,r}^{s-2})} \\
&\leq \nu \|\nabla(h_n - h_{n-1}) \nabla u_{n-1}\|_{\tilde{L}_t^2(B_{p,r}^{s-2})} + \nu \|\frac{h_n}{1+h_n} \nabla(h_n - h_{n-1}) \nabla u_{n-1}\|_{\tilde{L}_t^2(B_{p,r}^{s-2})} \\
&\leq \nu \|\nabla(h_n - h_{n-1}) \nabla u_{n-1}\|_{\tilde{L}_t^2(B_{p,r}^{s-2})} + \nu \|T_{\frac{h_n}{1+h_n} \nabla u_{n-1}} \nabla(h_n - h_{n-1})\|_{\tilde{L}_t^2(B_{p,r}^{s-2})} \\
&\quad + \nu \|T_{\nabla(h_n - h_{n-1})} \frac{h_n}{1+h_n} \nabla u_{n-1}\|_{\tilde{L}_t^2(B_{p,r}^{s-2})} + \nu \|R(\nabla(h_n - h_{n-1}), \frac{h_n}{1+h_n} \nabla u_{n-1})\|_{\tilde{L}_t^2(B_{p,r}^{s-2})} \\
&= I_{41} + I_{42} + I_{43} + I_{44}.
\end{aligned}$$

Set  $t^{s'} = \max\{t^{\frac{1}{2}}, t^{\frac{s-1}{2}}\}$ . Similar to the argument in the proof of  $I_3$ , we obtain

$$(3.18) \quad I_{41} \leq C E_1 t^{s'} \|h_n - h_{n-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})}.$$

Following the procedure of  $I_{31} - I_{33}$  respectively, we have

$$(3.19) \quad I_{42} \leq C E_1 E_2 t^{s'} \|h_n - h_{n-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})},$$

$$(3.20) \quad I_{43} \leq C E_1 E_2 t^{s'} \|h_n - h_{n-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})},$$

$$(3.21) \quad I_{44} \leq C E_1 E_2 t^{s'} \|h_n - h_{n-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})}.$$

Similarly as  $I_4$ , we have

$$\begin{aligned}
(3.22) \quad I_5 &= \|\nu \frac{h_n - h_{n-1}}{(1+h_n)(1+h_{n-1})} \nabla h_{n-1} \nabla u_{n-1}\|_{\tilde{L}_t^2(B_{p,r}^{s-2})} \\
&\leq C E_1 E_2 (1 + E_2)^2 t^{s'} \|h_n - h_{n-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})}.
\end{aligned}$$

We also have

$$(3.23) \quad \|\Delta_{n+1} u_0\|_{B_{p,r}^{s-1}} \leq 2^{-(n+1)} \|\Delta_{n+1} u_0\|_{B_{p,r}^s} \leq 2^{-(n+1)} \|u_0\|_{B_{p,r}^s}.$$

For  $h_{n+1} - h_n$ , we have

$$\begin{aligned}
(3.24) \quad & \|h_{n+1} - h_n\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \leq \exp(C_0 \int_0^t \|\nabla u_n\|_{B_{p,r}^{s-1}}) \times \\
& (\|\Delta_{n+1} h_0\|_{B_{p,r}^{s-1}} + \|\sum_{j=1}^4 J_j\|_{\tilde{L}_t^1(B_{p,r}^{s-1})}) \\
& \leq 2(\|\Delta_{n+1} h_0\|_{B_{p,r}^{s-1}} + t\|J_1 + J_4\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} + t^{\frac{1}{2}}\|J_2 + J_3\|_{\tilde{L}_t^2(B_{p,r}^{s-1})}).
\end{aligned}$$

From Lemmas 2.5-2.6, Lemma 2.8 and Lemma 2.15, we have

$$\begin{aligned}
(3.25) \quad & \|J_1\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} = \|(u_n - u_{n-1}) \cdot \nabla h_n\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \\
& \leq \|u_n - u_{n-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \|\nabla h_n\|_{L_t^\infty(L^\infty)} + \|\nabla h_n\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \|u_n - u_{n-1}\|_{L_t^\infty(L^\infty)} \\
& \leq C\|h_n\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \|u_n - u_{n-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \\
& \leq CE_2 \|u_n - u_{n-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})}.
\end{aligned}$$

In view of Lemma 2.6, we get

$$\begin{aligned}
(3.26) \quad & \|J_2\|_{\tilde{L}_t^2(B_{p,r}^{s-1})} = \|\operatorname{div}(u_n - u_{n-1})\|_{\tilde{L}_t^2(B_{p,r}^{s-1})} \\
& \leq C\|u_n - u_{n-1}\|_{\tilde{L}_t^2(B_{p,r}^s)}.
\end{aligned}$$

By Lemmas 2.5-2.6, Lemma 2.8 and Lemma 2.15, we obtain

$$\begin{aligned}
(3.27) \quad & \|J_3\|_{\tilde{L}_t^2(B_{p,r}^{s-1})} = \|h_n \operatorname{div}(u_n - u_{n-1})\|_{\tilde{L}_t^2(B_{p,r}^{s-1})} \\
& \leq \|h_n\|_{L_t^\infty(L^\infty)} \|\operatorname{div}(u_n - u_{n-1})\|_{\tilde{L}_t^2(B_{p,r}^{s-1})} + \|\operatorname{div}(u_n - u_{n-1})\|_{L_t^2(L^\infty)} \|h_n\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \\
& \leq C\|h_n\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \|u_n - u_{n-1}\|_{\tilde{L}_t^2(B_{p,r}^s)} \\
& \leq CE_2 \|u_n - u_{n-1}\|_{\tilde{L}_t^2(B_{p,r}^s)},
\end{aligned}$$

Similarly as  $J_3$ , we have

$$\begin{aligned}
(3.28) \quad & \|J_4\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} = \|(h_n - h_{n-1}) \operatorname{div} u_{n-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \\
& \leq \|h_n - h_{n-1}\|_{L_t^\infty(L^\infty)} \|\operatorname{div} u_{n-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} + \|\operatorname{div} u_{n-1}\|_{L_t^\infty(L^\infty)} \|h_n - h_{n-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \\
& \leq C\|h_n - h_{n-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \|u_{n-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \\
& \leq CE_1 \|h_n - h_{n-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})}.
\end{aligned}$$

We also have

$$(3.29) \quad \|\Delta_{n+1} h_0\|_{B_{p,r}^{s-1}} \leq 2^{-(n+1)} \|\Delta_{n+1} h_0\|_{B_{p,r}^s} \leq 2^{-(n+1)} \|h_0\|_{B_{p,r}^s}.$$

Moreover, by Lemma 2.14, we have

$$\begin{aligned}
(3.30) \quad & \|u_{n+1} - u_n\|_{\tilde{L}_t^2(B_{p,r}^s)} \leq \nu^{-\frac{1}{2}} C_0 e^{C_0(1+\nu t)^{\frac{1}{2}} \int_0^t \|\nabla u_n\|_{B_{p,r}^{s-1}} dt'} \times \\
& \left( (1+\nu t)^{\frac{1}{2}} \|\Delta_{n+1} u_0\|_{B_{p,r}^{s-1}} + (1+\nu t) \nu^{-\frac{1}{2}} \|\sum_{j=1}^5 F_j\|_{\tilde{L}_t^2(B_{p,r}^{s-2})} \right) \\
& \leq 4\nu^{-\frac{1}{2}} C_0 \|\Delta_{n+1} u_0\|_{B_{p,r}^{s-1}} + C \|\sum_{j=1}^5 F_j\|_{\tilde{L}_t^2(B_{p,r}^{s-2})}.
\end{aligned}$$



Combining (3.7)-(3.30), we get

$$\begin{aligned}
(3.31) \quad & \|u_{n+1} - u_n\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \leq C_0 2^{-n} \|u_0\|_{B_{p,r}^s} \\
& + C t^{s'} \left( E_2 \|u_n - u_{n-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} + E_2 \|u_n - u_{n-1}\|_{\tilde{L}_t^2(B_{p,r}^s)} \right. \\
& \left. + (1 + E_1(1 + E_2) + E_1 E_2(1 + E_2)^2) \|h_n - h_{n-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \right),
\end{aligned}$$

$$\begin{aligned}
(3.32) \quad & \|u_{n+1} - u_n\|_{\tilde{L}_t^2(B_{p,r}^s)} \leq 2\nu^{-\frac{1}{2}} C_0 2^{-n} \|u_0\|_{B_{p,r}^s} \\
& + C t^{s'} \left( E_2 \|u_n - u_{n-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} + E_2 \|u_n - u_{n-1}\|_{\tilde{L}_t^2(B_{p,r}^s)} \right. \\
& \left. + (1 + E_1(1 + E_2) + E_1 E_2(1 + E_2)^2) \|h_n - h_{n-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \right)
\end{aligned}$$

and

$$\begin{aligned}
(3.33) \quad & \|h_{n+1} - h_n\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \leq 2^{-n} \|h_0\|_{B_{p,r}^s} \\
& + C t \left( E_2 \|u_n - u_{n-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} + E_1 \|h_n - h_{n-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \right) \\
& + C t^{\frac{1}{2}} (1 + E_2) \|u_n - u_{n-1}\|_{\tilde{L}_t^2(B_{p,r}^s)}.
\end{aligned}$$

Choose a suitable  $T_2 (\leq T_1)$  such that:

$$(3.34) \quad \begin{cases} C E_2 T_2^{s'} \leq \frac{1}{12}, \\ C(1 + E_1(1 + E_2)^2 + E_1 E_2(1 + E_2)^2) T_2^{s'} \leq \frac{1}{12}, \\ C E_1 T_2 \leq \frac{1}{12}, \quad C E_2 T_2 \leq \frac{1}{12}, \\ C(1 + E_2) T_2^{\frac{1}{2}} \leq \frac{1}{12}. \end{cases}$$

Thus, for any  $t \leq T_2$ , we can obtain

$$\begin{aligned}
(3.35) \quad & \|u_{n+1} - u_n\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \leq \frac{1}{4} 2^{-n} E_1 \\
& + \frac{1}{12} \|u_n - u_{n-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} + \frac{1}{12} \|u_n - u_{n-1}\|_{\tilde{L}_t^2(B_{p,r}^s)} + \frac{1}{12} \|h_n - h_{n-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})},
\end{aligned}$$

$$\begin{aligned}
(3.36) \quad & \|u_{n+1} - u_n\|_{\tilde{L}_t^2(B_{p,r}^s)} \leq \frac{1}{2} \nu^{-1} 2^{-n} E_1 \\
& + \frac{1}{12} \|u_n - u_{n-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} + \frac{1}{12} \|u_n - u_{n-1}\|_{\tilde{L}_t^2(B_{p,r}^s)} + \frac{1}{12} \|h_n - h_{n-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})},
\end{aligned}$$

and

$$\begin{aligned}
(3.37) \quad & \|h_{n+1} - h_n\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \leq \frac{1}{4} 2^{-n} E_2 \\
& + \frac{1}{12} \|u_n - u_{n-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} + \frac{1}{12} \|u_n - u_{n-1}\|_{\tilde{L}_t^2(B_{p,r}^s)} + \frac{1}{12} \|h_n - h_{n-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})}.
\end{aligned}$$

We will temporarily assume that, for any  $k \leq n$

$$(3.38) \quad \|u_k - u_{k-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} + \|u_k - u_{k-1}\|_{\tilde{L}_t^2(B_{p,r}^s)} + \|h_k - h_{k-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \leq 4\nu^{-1} 2^{-k} (E_1 + E_2).$$

Then

$$\begin{aligned}
(3.39) \quad & \|u_{n+1} - u_n\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} + \|u_{n+1} - u_n\|_{\tilde{L}_t^2(B_{p,r}^s)} + \|h_{n+1} - h_n\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \\
& \leq \frac{1}{4}2^{-n}(E_1 + E_2) + \frac{1}{2}\nu^{-1}2^{-n}E_1 \\
& + \frac{1}{4}\|u_n - u_{n-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} + \|u_n - u_{n-1}\|_{\tilde{L}_t^2(B_{p,r}^s)} + \|h_n - h_{n-1}\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \\
& \leq 4\nu^{-1}2^{-n-1}(E_1 + E_2).
\end{aligned}$$

In order to complete the proof of Proposition 3.3, we only need justify the inequalities (3.38) hold for  $k = 1$ . It is obvious that

$$\begin{aligned}
(3.40) \quad & \|u_1 - u_0\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} + \|u_1 - u_0\|_{\tilde{L}_t^2(B_{p,r}^s)} + \|h_1 - h_0\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \\
& \leq 4(\|u_0\|_{B_{p,r}^s} + \|h_0\|_{B_{p,r}^s}) \\
& \leq 2\nu^{-1}(E_1 + E_2).
\end{aligned}$$

This complete the proof of Proposition 3.3.

### 3.3. Existence and uniqueness of local solution

In this subsection, we investigate the uniqueness of the local solution to the system (3.1). By Proposition 3.3, the approximative sequence  $(u_n, h_n)$  of the problem (2.1) is a Cauchy sequence in  $\chi([0, T], s-1, p, r, E_1, E_2)$  with  $s > \frac{d}{p} + 2$ , or  $s = \frac{d}{p} + 2, r = 1$ . So the limit  $(u, h)$  is a solution of the Cauchy problem (2.1). From Proposition 3.1, we obtain that this sequence is bounded in  $\chi([0, T], s, p, r, E_1, E_2)$ . So it's also the Cauchy sequence in  $\chi([0, T], s', p, r, E_1, E_2)$  for all  $s' < s$  by interpolation, and the limit is in  $\chi([0, T], s, p, r, E_1, E_2)$ . Thus we have proved local existence result in Theorem 1.1.

For the uniqueness result in Theorem 1.1, let  $(u, h)$  and  $(v, g)$  satisfy the problem (2.1) with the initial data  $(u_0, h_0), (v_0, g_0) \in B_{p,r}^s \times B_{p,r}^s$  respectively. Then we have

$$(3.41) \quad \begin{cases} \partial_t(u - v) + u \cdot \nabla(u - v) - \nu \Delta(u - v) = G_1(u, h) - G_1(v, g), \\ \partial_t(h - g) + u \cdot \nabla(h - g) = (u - v) \nabla g + G_2(u, h) - G_2(v, g), \\ (u - v)|_{t=0} = 0, (h - g)|_{t=0} = 0. \end{cases}$$

Using Lemmas 2.13-2.14, we can get

$$\begin{aligned}
(3.42) \quad & \|u - v\|_{\tilde{L}^\infty(B_{p,r}^s)} + \|h - g\|_{\tilde{L}^\infty(B_{p,r}^s)} \\
& \leq Ce^{Ct}(\|u_0 - v_0\|_{B_{p,r}^s} + \|h_0 - g_0\|_{B_{p,r}^s}) + C_1 t \|u - v\|_{\tilde{L}^\infty(B_{p,r}^s)} + C_2 t \|h - g\|_{\tilde{L}^\infty(B_{p,r}^s)}.
\end{aligned}$$

This gives the uniqueness of Theorem 1.1.

### 3.4. Continuity

In this subsection, we will prove that  $u, h \in \mathcal{C}([0, T]; B_{p,r}^s)$ . From the equations we can get  $u, h \in C([0, T]; B_{p,r}^{s-2})$ . Then we have that  $\Delta_j u, \Delta_j h \in \mathcal{C}([0, T]; B_{p,r}^s)$  for any  $j \geq -1$ , from which it follows that  $S_j u, S_j h \in \mathcal{C}([0, T]; B_{p,r}^s)$  for all  $j \in \mathbb{N}$ . We claim that the sequence of continuous  $B_{p,r}^s$ -valued functions  $\{S_j f\}_{j \in \mathbb{N}}$  converges uniformly on  $[0, T]$ . Indeed, by Proposition 2.1, we have

$$\Delta_{j'}(u - S_j u) = \sum_{|j' - j''| \leq 1, j'' \geq j} \Delta_{j'} \Delta_{j''} u, \quad \Delta_{j'}(h - S_j h) = \sum_{|j' - j''| \leq 1, j'' \geq j} \Delta_{j'} \Delta_{j''} h,$$

from which it follows that

$$(3.43) \quad \begin{aligned} \|u - S_j u\|_{\tilde{L}^\infty(B_{p,r}^s)} &\leq C \left( \sum_{j' \geq j-1} 2^{j' sr} \|\Delta_{j'} u\|_{L^\infty(L^p)}^r \right)^{\frac{1}{r}}, \\ \|h - S_j h\|_{\tilde{L}^\infty(B_{p,r}^s)} &\leq C \left( \sum_{j' \geq j-1} 2^{j' sr} \|\Delta_{j'} h\|_{L^\infty(L^p)}^r \right)^{\frac{1}{r}}. \end{aligned}$$

Applying the operator  $\Delta_{j'}$  in (2.1), we get

$$(3.44) \quad \begin{cases} \partial_t \Delta_{j'} u + \Delta_{j'}((u \cdot \nabla)u) - \nu \Delta_{j'}(\Delta u) - \nu \Delta_{j'}(\nu \nabla(\ln(1+h)) \nabla u) + \Delta_{j'}(\nabla h) = 0, \\ \partial_t \Delta_{j'} h + \Delta_{j'}(\operatorname{div} u) + \Delta_{j'}(h \operatorname{div} u) + \Delta_{j'}(u \cdot \nabla h) = 0, \\ \Delta_{j'} u|_{t=0} = \Delta_{j'} u_0, \quad \Delta_{j'} h|_{t=0} = \Delta_{j'} h_0. \end{cases}$$

When  $j \geq 1$ , we have the Fourier transform of  $\Delta_{j'} u_0$  and  $\Delta_{j'} f$  is supported in an annulus  $2^{j'} \mathcal{C}$ , by Lemma 2.17, for  $\rho > 2$ , we have

$$(3.45) \quad \|\Delta_{j'} u\|_{L^\infty(L^p)} \leq C \|\Delta_{j'} u_0\|_{L^p} + 2^{2j'(-1+\frac{1}{\rho})} \|\Delta_{j'} f\|_{L_T^\rho(L^p)},$$

where  $f = -\nabla h - u \cdot \nabla u + \nu \nabla(\ln(1+h)) \nabla u$ . By Lemma 2.15, we have

$$(3.46) \quad \begin{aligned} \|f\|_{\tilde{L}_T^\infty(B_{p,r}^{s-1})} &\leq \|\nabla h\|_{\tilde{L}_T^\infty(B_{p,r}^{s-1})} + \|u \cdot \nabla u\|_{\tilde{L}_T^\infty(B_{p,r}^{s-1})} + \nu \|\nabla(\ln(1+h)) \nabla u\|_{\tilde{L}_T^\infty(B_{p,r}^{s-1})} \\ &\leq C \|h\|_{\tilde{L}_T^\infty(B_{p,r}^s)} + C \|u\|_{L_T^\infty(L^\infty)} \|\nabla u\|_{\tilde{L}_T^\infty(B_{p,r}^{s-1})} + C \|\nabla u\|_{L_T^\infty(L^\infty)} \|u\|_{\tilde{L}_T^\infty(B_{p,r}^{s-1})} + \\ &\quad C \|\nabla(\ln(1+h))\|_{L_T^\infty(L^\infty)} \|\nabla u\|_{\tilde{L}_T^\infty(B_{p,r}^{s-1})} + C \|\nabla u\|_{L_T^\infty(L^\infty)} \|\nabla(\ln(1+h))\|_{\tilde{L}_T^\infty(B_{p,r}^{s-1})} \\ &\leq C \|h\|_{\tilde{L}_T^\infty(B_{p,r}^s)} + C \|h\|_{\tilde{L}_T^\infty(B_{p,r}^s)} \|u\|_{\tilde{L}_T^\infty(B_{p,r}^s)}. \end{aligned}$$

Thus, by (3.45) and (3.46), for  $\rho > 2$ , we have

$$(3.47) \quad \begin{aligned} \|u - S_j u\|_{\tilde{L}_T^\infty(B_{p,r}^s)}^r &\leq C \sum_{j' \geq j-1} 2^{j' sr} \|\Delta_{j'} u\|_{L_T^\infty(L^p)}^r \\ &\leq \sum_{j' \geq j-1} 2^{j' sr} \left( \|\Delta_{j'} u_0\|_{L^p} + 2^{2j'(-1+\frac{1}{\rho})} \|\Delta_{j'} f\|_{L_T^\rho(L^p)} \right)^r \\ &\leq C \sum_{j' \geq j-1} 2^{j' sr} \|\Delta_{j'} u_0\|_{L^p}^r + C \sum_{j' \geq j-1} \left( 2^{j'(s-2+\frac{2}{\rho})} \|\Delta_{j'} f\|_{L_T^\rho(L^p)} \right)^r \\ &\leq C \sum_{j' \geq j-1} 2^{j' sr} \|\Delta_{j'} u_0\|_{L^p}^r + C \sum_{j' \geq j-1} \left( 2^{j'(\frac{2}{\rho}-1)} \left( \int_0^T 2^{j'(s-1)} \|\Delta_{j'} f\|_{L^p}^\rho dt \right)^{\frac{1}{\rho}} \right)^r \\ &\leq C \sum_{j' \geq j-1} 2^{j' sr} \|\Delta_{j'} u_0\|_{L^p}^r + C \|f\|_{\tilde{L}_T^\infty(B_{p,r}^{s-1})}^r \sum_{j' \geq j-1} \left( 2^{j'(2-\rho)} \int_0^T d_{j'}^\rho(t) dt \right)^{\frac{r}{\rho}}, \end{aligned}$$

where  $d_{j'}(t) \in \ell^r$  and  $\|d_{j'}(t)\|_{\ell^r} = 1$ .

The first term clearly tends to 0 when  $j$  goes to  $\infty$ . For the second term, when  $\rho \leq r$ , we have

$$\begin{aligned}
(3.48) \quad & \sum_{j' \geq j-1} (2^{j'(2-\rho)} \int_0^T d_{j'}^\rho(t) dt)^{\frac{r}{\rho}} \\
& \leq C \sum_{j' \geq j-1} (T^{\frac{r-\rho}{r}} (\int_0^T d_{j'}^r(t) dt)^{\frac{\rho}{r}})^{\frac{r}{\rho}} \\
& \leq CT^{\frac{r-\rho}{\rho}} \int_0^T \sum_{j' \geq j-1} d_{j'}^r(t) dt.
\end{aligned}$$

By virtue of Lebesgue's dominated convergence theorem, for  $\rho \leq r$ , the second term tends 0 when  $j$  goes to  $\infty$ . As regards  $\rho > r$ , by Yong's inequality, we have

$$\begin{aligned}
(3.49) \quad & \sum_{j' \geq j-1} (2^{j'(2-\rho)} \int_0^T d_{j'}^\rho(t) dt)^{\frac{r}{\rho}} \\
& \leq C \sum_{j' \geq j-1} ((2^{j'(2-\rho)})^{\frac{r}{\rho} \times \frac{\rho}{\rho-r}} + (\int_0^T d_{j'}^\rho(t) dt)^{\frac{r}{\rho} \times \frac{\rho}{r}}) \\
& \leq C \sum_{j' \geq j-1} 2^{j'(\frac{r(2-\rho)}{\rho-r})} + C \int_0^T \sum_{j' \geq j-1} d_{j'}^\rho(t) dt \\
& \leq C \sum_{j' \geq j-1} 2^{j'(2-\rho)\frac{r}{\rho-r}} + C \int_0^T \sum_{j' \geq j-1} d_{j'}^r(t) dt.
\end{aligned}$$

By virtue of Lebesgue's dominated convergence theorem and  $\rho > 2$ , for  $\rho > r$ , the second term tends 0 when  $j$  goes to  $\infty$  as well. This completes the proof of continuity for  $u$ .

As regards  $h$ , multiplying the second equation of (3.44) by  $\Delta_j h |\Delta_j h|^{p-2}$ , integrating over  $\mathbb{R}^2$  yields

$$\begin{aligned}
(3.50) \quad & \frac{1}{p} \partial_t \|\Delta_j h\|_{L^p}^p \leq \left| \int_{\mathbb{R}^2} \Delta_{j'}(\operatorname{div} u) \Delta_{j'} h |\Delta_{j'} h|^{p-2} dx \right| \\
& + \left| \int_{\mathbb{R}^2} \Delta_{j'}(h \operatorname{div} u) \Delta_{j'} h |\Delta_{j'} h|^{p-2} dx \right| + \left| \int_{\mathbb{R}^2} \Delta_{j'}((u \cdot)h) \Delta_{j'} h |\Delta_{j'} h|^{p-2} dx \right| \\
& = H_1 + H_2 + H_3.
\end{aligned}$$

Next, we deal with  $H_1 - H_3$  term by term. For  $H_1$ , it's easy to check that

$$\begin{aligned}
(3.51) \quad & H_1 = \int_{\mathbb{R}^2} \Delta_{j'}(\operatorname{div} u) \Delta_{j'} h |\Delta_{j'} h|^{p-2} dx \\
& \leq C \|\Delta_{j'} \operatorname{div} u\|_{L^p} \|\Delta_{j'} h\|_{L^p}^{p-1}.
\end{aligned}$$

Then we divide  $H_2$  to three terms

$$\begin{aligned}
(3.52) \quad & H_2 = \int_{\mathbb{R}^2} \Delta_{j'}(h \operatorname{div} u) \Delta_{j'} h |\Delta_{j'} h|^{p-2} dx \\
& = \int_{\mathbb{R}^2} \Delta_{j'}(T_h \operatorname{div} u + T_{\operatorname{div} u} h + R(\operatorname{div} u, h)) \Delta_{j'} h |\Delta_{j'} h|^{p-2} dx \\
& = H_{21} + H_{22} + H_{23}.
\end{aligned}$$

For  $H_{21}$ , by a simple computation, we get

$$\begin{aligned}
(3.53) \quad & H_{21} = \sum_{|j'-q| \leq 4} \int_{\mathbb{R}^2} \Delta_{j'}(S_q h \Delta_q(\operatorname{div} u)) \Delta_{j'} h |\Delta_{j'} h|^{p-2} dx \\
& \leq C \|h\|_{L^\infty} \sum_{|j'-q| \leq 4} \|\Delta_q(\operatorname{div} u)\|_{L^p} \|\Delta_{j'} h\|_{L^p}^{p-1}.
\end{aligned}$$

For  $H_{22}$  and  $H_{23}$ , by a discrete Young's inequality, we have

$$\begin{aligned}
H_{22} &= \sum_{|j'-q|\leq 4} \int_{\mathbb{R}^2} \Delta_{j'} (S_q(\operatorname{div} u) \Delta_q h) \Delta_{j'} h |\Delta_{j'} h|^{p-2} dx \\
&\leq \|\operatorname{div} u\|_{L^\infty} \sum_{|j'-q|\leq 4} \|\Delta_q h\|_{L^p} \|\Delta_{j'} h\|_{L^p}^{p-1} \\
&\leq \|\operatorname{div} u\|_{L^\infty} \sum_{|j'-q|\leq 4} d_q 2^{-qs} \|h\|_{B_{p,r}^s} \|\Delta_{j'} h\|_{L^p}^{p-1} \\
&\leq C \|\operatorname{div} u\|_{L^\infty} d_{j'}(t) 2^{-j's} \|h\|_{B_{p,r}^s} \|\Delta_{j'} h\|_{L^p}^{p-1},
\end{aligned} \tag{3.54}$$

and

$$\begin{aligned}
H_{23} &= \sum_{q \geq j'-N, |q-q'|\leq 1} \int_{\mathbb{R}^2} \Delta_{j'} (\Delta_{q'}(\operatorname{div} u) \Delta_q h) \Delta_{j'} h |\Delta_{j'} h|^{p-2} dx \\
&\leq C \|\operatorname{div} u\|_{L^\infty} \sum_{q \geq j'-N} \|\Delta_q h\|_{L^p} \|\Delta_{j'} h\|_{L^p}^{p-1} \\
&\leq \|\operatorname{div} u\|_{L^\infty} \sum_{q \geq j'-N} d_q 2^{-qs} \|h\|_{B_{p,r}^s} \|\Delta_{j'} h\|_{L^p}^{p-1} \\
&\leq C \|\operatorname{div} u\|_{L^\infty} d_{j'}(t) 2^{-j's} \|h\|_{B_{p,r}^s} \|\Delta_{j'} h\|_{L^p}^{p-1}.
\end{aligned} \tag{3.55}$$

For  $H_3$ , we can divide it into two parts

$$\begin{aligned}
H_3 &= \int_{\mathbb{R}^2} \Delta_{j'} ((u \cdot \nabla) h) \Delta_{j'} h |\Delta_{j'} h|^{p-2} dx \\
&= \int_{\mathbb{R}^2} (u \cdot \nabla) \Delta_{j'} h \Delta_{j'} h |\Delta_{j'} h|^{p-2} dx - \int_{\mathbb{R}^2} R_{j'}(u, h) \Delta_{j'} h |\Delta_{j'} h|^{p-2} dx \\
&= H_{31} + H_{32},
\end{aligned} \tag{3.56}$$

where the definition of  $R_{j'}(u, h)$  is the same as that in Lemma 2.17. For  $H_{31}$ , we obtain that

$$\begin{aligned}
H_{31} &= \int_{\mathbb{R}^2} (u \cdot \nabla) \Delta_{j'} h \Delta_{j'} h |\Delta_{j'} h|^{p-2} dx \\
&= \frac{1}{p} \int_{\mathbb{R}^2} u \cdot \nabla |\Delta_{j'} h|^p dx \\
&= -\frac{1}{p} \int_{\mathbb{R}^2} \operatorname{div} u |\Delta_{j'} h|^p dx \\
&\leq C \|\nabla u\|_{L^\infty} \|\Delta_{j'} h\|_{L^p}^p.
\end{aligned} \tag{3.57}$$

For  $H_{32}$ , by the second of Lemma 2.17 with  $p_2 = \infty$ ,  $p_1 = p$ , we have

$$\begin{aligned}
H_{32} &= \int_{\mathbb{R}^2} R_{j'}(u, h) \Delta_{j'} h |\Delta_{j'} h|^{p-2} dx \\
&\leq C \|R_{j'}(u, h)\|_{L^p} \|\Delta_{j'} h\|_{L^p}^{p-1} \\
&\leq C d_{j'}(t) 2^{-j's} (\|\nabla u\|_{L^\infty} \|h\|_{B_{p,r}^s} + \|\nabla h\|_{L^\infty} \|u\|_{B_{p,r}^s}) \|\Delta_{j'} h\|_{L^p}^{p-1}.
\end{aligned} \tag{3.58}$$

Combining (3.50)-(3.58), we get

$$\begin{aligned}
&\partial_t \|\Delta_{j'} h\|_{L^p}^p \leq C \left( \|\Delta_{j'} \operatorname{div} u\|_{L^p} \|\Delta_{j'} h\|_{L^p}^{p-1} + \|h\|_{L^\infty} \sum_{|j'-q|\leq 4} \|\Delta_q \operatorname{div} u\|_{L^p} \|\Delta_{j'} h\|_{L^p}^{p-1} \right. \\
&+ \|\operatorname{div} u\|_{L^\infty} d_{j'}(t) 2^{-j's} \|h\|_{B_{p,r}^s} \|\Delta_{j'} h\|_{L^p}^{p-1} + \|\nabla u\|_{L^\infty} \|\Delta_{j'} h\|_{L^p}^p \\
&\left. + 2^{-j's} d_{j'}(t) (\|\nabla u\|_{L^\infty} \|h\|_{B_{p,r}^s} + \|\nabla h\|_{L^\infty} \|u\|_{B_{p,r}^s}) \|\Delta_{j'} h\|_{L^p}^{p-1} \right),
\end{aligned} \tag{3.59}$$

where  $d_{j'}(t) \in \ell^r$ , and  $\|d_{j'}(t)\|_{\ell^r} = 1$ .

Then it follows

$$\begin{aligned}
(3.60) \quad & \partial_t \|\Delta_{j'} h\|_{L^p} \leq C \left( \|\Delta_{j'} \operatorname{div} u\|_{L^p} + \|h\|_{L^\infty} \sum_{|j'-q| \leq 4} \|\Delta_q \operatorname{div} u\|_{L^p} \right. \\
& + \|\operatorname{div} u\|_{L^\infty} d_{j'} 2^{-j's} \|h\|_{B_{p,r}^s} + \|\nabla u\|_{L^\infty} \|\Delta_{j'} h\|_{L^p} \\
& \left. + 2^{-j's} d_{j'}(t) (\|\nabla u\|_{L^\infty} \|h\|_{B_{p,r}^s} + \|\nabla h\|_{L^\infty} \|u\|_{B_{p,r}^s}) \right) \\
& \leq C \left( \|\Delta_{j'} \operatorname{div} u\|_{L^p} + \|h\|_{L^\infty} \sum_{|j'-q| \leq 4} \|\Delta_q \operatorname{div} u\|_{L^p} \right. \\
& \left. + 2^{-j's} d_{j'}(t) (\|\nabla u\|_{L^\infty} + \|\nabla h\|_{L^\infty}) (\|u\|_{B_{p,r}^s} + \|h\|_{B_{p,r}^s}) \right).
\end{aligned}$$

For any  $t \in (0, T]$ , integrating (3.60) from 0 to  $t$ , and taking  $r$  power, we get

$$\begin{aligned}
(3.61) \quad & \|\Delta_{j'} h(t)\|_{L^p}^r \leq C \|\Delta_{j'} h_0\|_{L^p}^r + C \left( \int_0^t 2^{-j's} d_{j'}(t') (\|\nabla u\|_{L^\infty} + \|\nabla h\|_{L^\infty}) (\|u\|_{B_{p,r}^s} + \|h\|_{B_{p,r}^s}) dt' \right)^r \\
& + C \left( \int_0^t (\|\Delta_{j'} \operatorname{div} u\|_{L^p} + \|h\|_{L^\infty} \sum_{|j'-q| \leq 4} \|\Delta_q \operatorname{div} u\|_{L^p}) dt' \right)^r.
\end{aligned}$$

By the Minkowski inequality, it follows that

$$\begin{aligned}
(3.62) \quad & \|h - S_j h\|_{\tilde{L}_T^\infty(B_{p,r}^s)} \leq C \left( \sum_{j' \geq j-1} 2^{j'sr} \|\Delta_{j'} h\|_{L_T^\infty(L^p)}^r \right)^{\frac{1}{r}} \\
& \leq C \left( \sum_{j' \geq j-1} 2^{j'sr} \|\Delta_{j'} h_0\|_{L^p}^r \right)^{\frac{1}{r}} + \\
& C \left( \sum_{j' \geq j-1} \left( \int_0^T d_{j'}(t) (\|\nabla u\|_{L^\infty} + \|\nabla h\|_{L^\infty}) (\|u\|_{B_{p,r}^s} + \|h\|_{B_{p,r}^s}) dt \right)^r \right)^{\frac{1}{r}} \\
& + C \left( \sum_{j' \geq j-1} 2^{j'sr} \left( \int_0^T (\|\Delta_{j'} \operatorname{div} u\|_{L^p} + \|h\|_{L^\infty} \sum_{|j'-q| \leq 4} \|\Delta_q \operatorname{div} u\|_{L^p}) dt \right)^r \right)^{\frac{1}{r}} \\
& \leq C \left( \sum_{j' \geq j-1} 2^{j'sr} \|\Delta_{j'} h_0\|_{L^p}^r \right)^{\frac{1}{r}} \\
& + C (\|u\|_{\tilde{L}_T^\infty(B_{p,r}^s)} + \|h\|_{\tilde{L}_T^\infty(B_{p,r}^s)})^2 \int_0^T \left( \sum_{j' \geq j-1} d_{j'}^r(t) \right)^{\frac{1}{r}} dt \\
& + C \left( \sum_{j' \geq j-1} 2^{j'sr} \left( \int_0^T (\|\Delta_{j'} \operatorname{div} u\|_{L^p} + \|h\|_{L^\infty} \sum_{|j'-q| \leq 4} \|\Delta_q \operatorname{div} u\|_{L^p}) dt \right)^r \right)^{\frac{1}{r}},
\end{aligned}$$

The first term and the second term clearly tends to 0 when  $j$  goes to  $\infty$ . For the third term, by the Young inequality, we only need to deal with

$$C \left( \sum_{j' \geq j-1} (2^{j's} \int_0^T \|\Delta_{j'} \operatorname{div} u\|_{L^p} dt)^r \right)^{\frac{1}{r}},$$

it tends to 0 when  $j$  goes to  $\infty$  provided  $\|\operatorname{div} u\|_{\tilde{L}_T^1(B_{p,r}^s)}$  is bounded. Actually, by Lemma 2.14, we

have

$$\begin{aligned}
& \|u\|_{\tilde{L}_T^1(B_{p,r}^{s+1})} \leq C\nu^{-1} \exp\left(C(1+\nu T) \int_0^T \|\nabla u\|_{L^\infty} dt\right) \times \\
& \left( (1+\nu T)\|u_0\|_{B_{p,r}^{s-1}} + (1+\nu T)\|\nabla h + \nabla u \nabla(\ln(1+h))\|_{\tilde{L}_T^1(B_{p,r}^{s-1})} \right) \\
(3.63) \quad & \leq C_T \left( \|u_0\|_{B_{p,r}^s} + \|h\|_{\tilde{L}_T^1(B_{p,r}^s)} + \|\nabla u\|_{L_T^\infty(L^\infty)} \|\nabla(\ln(1+h))\|_{\tilde{L}_T^1(B_{p,r}^{s-1})} \right. \\
& \left. + \|\nabla(\ln(1+h))\|_{L_T^\infty(L^\infty)} \|\nabla u\|_{\tilde{L}_T^1(B_{p,r}^{s-1})} \right) \\
& \leq C_T \left( \|u_0\|_{B_{p,r}^s} + T\|h\|_{\tilde{L}_T^\infty(B_{p,r}^s)} + T\|u\|_{\tilde{L}_T^\infty(B_{p,r}^s)} \|h\|_{\tilde{L}_T^\infty(B_{p,r}^s)} \right).
\end{aligned}$$

This completes the proof of continuity for  $h$ .

## 4 Blow-up criteria and Global existence

In this section, we will present a blow-up criterion in  $B_{p,r}^s$ ,  $s > \frac{2}{p} + 1$  and get the global existence of the system with small enough initial data in  $B_{p,r}^s(\mathbb{R}^2)$ ,  $p \leq 2$  and  $s > 1 + \frac{2}{p}$ . Next we divide the section into three subsections.

### 4.1. A priori estimate

In this subsection, we give a priori estimate as follows:

**Proposition 4.1.** *Let  $u_0, h_0 \in B_{p,r}^{s+\varepsilon} \times B_{p,r}^{s+\varepsilon}$ , where  $\varepsilon$  is a small enough positive real number when  $r > 2$ , and  $\varepsilon = 0$  when  $r \leq 2$ , and let  $(u, h)$  be the corresponding solution of Cauchy problem (2.1) in  $B_{p,r}^{s+\varepsilon} \times B_{p,r}^{s+\varepsilon}$ . Assume that  $T^*$  is the maximal existence time of the solution, and*

$$\int_0^{T^*} \|\nabla u\|_{L^\infty}^2 + \|h\|_{L^\infty}^2 + \|\nabla(\ln(1+h))\|_{L^\infty}^2 dt < C_{T^*}.$$

Then we have

$$\|u\|_{\tilde{L}_{T^*}^\infty(B_{p,2}^s)} + \|h\|_{\tilde{L}_{T^*}^\infty(B_{p,2}^s)} \leq C(s, p, r, \nu, C_{T^*}, \|u_0\|_{B_{p,r}^{s+\varepsilon}}, \|h_0\|_{B_{p,r}^{s+\varepsilon}}).$$

Proof: By Lemma 2.13, for any  $T < T^*$  we have

$$\begin{aligned}
& \|u\|_{\tilde{L}_T^\infty(B_{p,2}^s)} \leq C e^{\int_0^T \|\nabla u\|_{L^\infty} dt} \left( \|u_0\|_{B_{p,2}^s} + \left(\frac{1+\nu T}{\nu}\right)^{\frac{1}{2}} \|\nabla h + \nabla u \nabla(\ln(1+h))\|_{\tilde{L}_T^2(B_{p,2}^{s-1})} \right) \\
(4.1) \quad & \leq C_T \left( \|u_0\|_{B_{p,2}^s} + \|h\|_{\tilde{L}_T^2(B_{p,2}^s)} + \|\nabla u \nabla(\ln(1+h))\|_{L_T^2(B_{p,2}^{s-1})} \right) \\
& \leq C_T \left( \|u_0\|_{B_{p,2}^s} + \|h\|_{\tilde{L}_T^2(B_{p,2}^s)} + \right. \\
& \left. \left( \int_0^T (\|\nabla u\|_{L^\infty}^2 + \|\nabla(\ln(1+h))\|_{L^\infty}^2) (\|u\|_{B_{p,2}^s}^2 + \|h\|_{B_{p,2}^s}^2) dt \right)^{\frac{1}{2}} \right).
\end{aligned}$$

From (3.66), we have

$$\begin{aligned}
(4.2) \quad & \partial_t \|\Delta_j h\|_{L^p}^2 \leq C \left( \|\Delta_j \operatorname{div} u\|_{L^p} \|\Delta_j h\|_{L^p} + \|h\|_{L^\infty} \sum_{|j-q| \leq 4} \|\Delta_q \operatorname{div} u\|_{L^p} \|\Delta_j h\|_{L^p} \right. \\
& + \|\operatorname{div} u\|_{L^\infty} d_j(t) 2^{-js} \|h\|_{B_{p,2}^s} \|\Delta_j h\|_{L^p} + \|\nabla u\|_{L^\infty} \|\Delta_j h\|_{L^p}^2 \\
& \left. + 2^{-js} d_j(t) (\|\nabla u\|_{L^\infty} \|h\|_{B_{p,2}^s} + \|\nabla h\|_{L^\infty} \|u\|_{B_{p,2}^s}) \|\Delta_j h\|_{L^p} \right) \\
& \leq C(1 + \|h\|_{L^\infty}^2) \|\Delta_j h\|_{L^p}^2 + C \|\Delta_j \operatorname{div} u\|_{L^p}^2 + C(\sum_{|j-q| \leq 4} \|\Delta_q \operatorname{div} u\|_{L^p})^2 \\
& + C(\|\operatorname{div} u\|_{L^\infty} d_j(t) 2^{-js} \|h\|_{B_{p,2}^s} \|\Delta_j h\|_{L^p} + \|\nabla u\|_{L^\infty} \|\Delta_j h\|_{L^p}^2 \\
& + 2^{-js} d_j(t) (\|\nabla u\|_{L^\infty} \|h\|_{B_{p,2}^s} + \|\nabla h\|_{L^\infty} \|u\|_{B_{p,2}^s}) \|\Delta_j h\|_{L^p}) \\
& \leq C(1 + \|h\|_{L^\infty}^2 + \|\nabla h\|_{L^\infty}^2 + \|\nabla u\|_{L^\infty}^2) 2^{-2js} d_j^2(t) (\|u\|_{B_{p,2}^s}^2 + \|h\|_{B_{p,2}^s}^2) \\
& + C \|\Delta_j \operatorname{div} u\|_{L^p}^2 + C(\sum_{|j-q| \leq 4} \|\Delta_q \operatorname{div} u\|_{L^p})^2.
\end{aligned}$$

Then it follows that

$$\begin{aligned}
(4.3) \quad & \|\Delta_j h\|_{L^p}^2 \leq \|\Delta_j h_0\|_{L^p}^2 + C \int_0^t (\|\Delta_j \operatorname{div} u\|_{L^p}^2 + (\sum_{|j-q| \leq 4} \|\Delta_q \operatorname{div} u\|_{L^p})^2) dt' \\
& + C \int_0^t (1 + \|h\|_{L^\infty}^2 + \|\nabla h\|_{L^\infty}^2 + \|\nabla u\|_{L^\infty}^2) 2^{-2js} d_j^2(t) (\|u\|_{B_{p,2}^s}^2 + \|h\|_{B_{p,2}^s}^2) dt'.
\end{aligned}$$

Multiplying by  $2^{2js}$  and taking the sum over  $j$  from  $-1$  to  $\infty$  thus give

$$\begin{aligned}
(4.4) \quad & \|h\|_{\tilde{L}_T^2(B_{p,2}^s)}^2 \leq \|h_0\|_{B_{p,2}^s}^2 + C \sum_{j \geq -1} 2^{2js} \int_0^t (\|\Delta_j \operatorname{div} u\|_{L^p}^2 + (\sum_{|j-q| \leq 4} \|\Delta_q \operatorname{div} u\|_{L^p})^2) dt' \\
& + C \sum_{j \geq -1} \int_0^t d_j^2(t') (1 + \|h\|_{L^\infty}^2 + \|\nabla u\|_{L^\infty}^2 + \|\nabla h\|_{L^\infty}^2) (\|u\|_{B_{p,2}^s}^2 + \|h\|_{B_{p,2}^s}^2) dt' \\
& \leq \|h_0\|_{B_{p,2}^s}^2 + C \|\operatorname{div} u\|_{\tilde{L}_T^2(B_{p,2}^s)}^2 \\
& + C \int_0^T (1 + \|h\|_{L^\infty}^2 + \|\nabla u\|_{L^\infty}^2 + \|\nabla h\|_{L^\infty}^2) (\|u\|_{B_{p,2}^s}^2 + \|h\|_{B_{p,2}^s}^2) dt'.
\end{aligned}$$

By Lemma 2.13 again, we have

$$\begin{aligned}
(4.5) \quad & \|\operatorname{div} u\|_{\tilde{L}_T^2(B_{p,2}^s)} \leq C \nu^{-\frac{1}{2}} e^{C(1+\nu T)^{\frac{1}{2}}} \int_0^T \|\nabla u\|_{L^\infty} \times \\
& ((1 + \nu T)^{\frac{1}{2}} \|u_0\|_{B_{p,2}^s} + (1 + \nu T)^{\frac{1}{2}} \nu^{-\frac{1}{2}} \|\nabla h + \nabla(\ln(1+h))\nabla u\|_{\tilde{L}_T^2(B_{p,2}^{s-1})}) \\
& \leq C_T (\|u_0\|_{B_{p,2}^s} + \|h\|_{L_T^2(B_{p,2}^s)} + \|\nabla u \nabla(\ln(1+h))\|_{L_T^2(B_{p,2}^{s-1})}) \\
& \leq C_T \left( \|u_0\|_{B_{p,2}^s} + \|h\|_{L_T^2(B_{p,2}^s)} + \left( \int_0^T (\|\nabla u\|_{L^\infty}^2 + \|\nabla(\ln(1+h))\|_{L^\infty}^2) (\|u\|_{B_{p,2}^s}^2 + \|h\|_{B_{p,2}^s}^2) dt \right)^{\frac{1}{2}} \right).
\end{aligned}$$

Combining (4.1), (4.4) and (4.5), and noting that

$$\|\nabla h\|_{L^\infty} \leq (1 + \|h\|_{L^\infty}) \|\nabla(\ln(1+h))\|_{L^\infty},$$



we get

$$(4.6) \quad \begin{aligned} & \|u\|_{\tilde{L}_T^\infty(B_{p,2}^s)}^2 + \|h\|_{\tilde{L}_T^\infty(B_{p,2}^s)}^2 \leq C_T (\|u_0\|_{B_{p,2}^s}^2 + \|h_0\|_{B_{p,2}^s}^2) \\ & + C_T \int_0^T (1 + \|h\|_{L^\infty}^2 + \|\nabla(\ln(1+h))\|_{L^\infty}^2 + \|\nabla u\|_{L^\infty}^2) (\|u\|_{\tilde{L}_T^\infty(B_{p,2}^s)}^2 + \|h\|_{\tilde{L}_T^\infty(B_{p,2}^s)}^2) dt. \end{aligned}$$

By the viture of Gronwall's inequality, we can obtain

$$(4.7) \quad \|u\|_{\tilde{L}_T^\infty(B_{p,2}^s)}^2 + \|h\|_{\tilde{L}_T^\infty(B_{p,2}^s)}^2 \leq C_T (\|u_0\|_{B_{p,2}^s}^2 + \|h_0\|_{B_{p,2}^s}^2) e^{C_T \int_0^T (1 + \|h\|_{L^\infty}^2 + \|\nabla(\ln(1+h))\|_{L^\infty}^2 + \|\nabla u\|_{L^\infty}^2) dt}.$$

#### 4.2. Blow-up criterion in $B_{p,r}^s$

In this subsection, we establish a blow-up criterion in common  $B_{p,r}^s$  with  $s > 1 + \frac{2}{p}$ .

**Proposition 4.2.** *Let  $u_0, h_0 \in B_{p,r}^s \times B_{p,r}^s$ ,  $s > 1 + \frac{2}{p}$ , and let  $(u, h)$  be the corresponding solution of the Cauchy problem (2.1) in  $B_{p,r}^s \times B_{p,r}^s$ . Assume that  $T^*$  is the maximal existence time of solution. If  $T^*$  is finite, then we have,*

$$\int_0^{T^*} \|\nabla u\|_{L^\infty}^{r_1} + \|h\|_{L^\infty}^{r_1} + \|\nabla(\ln(1+h))\|_{L^\infty}^{r_1} dt' = \infty,$$

where  $r_1 = \max\{r', 2\}$ .

Proof: If

$$\int_0^{T^*} \|\nabla u\|_{L^\infty}^{r_1} + \|h\|_{L^\infty}^{r_1} + \|\nabla(\ln(1+h))\|_{L^\infty}^{r_1} dt' < \infty,$$

in view of Proposition 4.1, then we have

$$\|u\|_{\tilde{L}_{T^*}^\infty(B_{p,2}^{s-\varepsilon})} + \|h\|_{\tilde{L}_{T^*}^\infty(B_{p,2}^{s-\varepsilon})} < C_{T^*},$$

here  $\varepsilon = 0$  when  $r \leq 2$  and  $\varepsilon$  is a small enough positive real number when  $r > 2$ . Then we take a suitable  $\rho \geq 2$  such that  $B_{p,r}^{s-1+\frac{2}{\rho}}$  is continously embedded  $B_{p,2}^{s-\varepsilon}$ , thus by Lemma 2.14-2.15, for any  $t < T^*$ , we have

$$(4.8) \quad \begin{aligned} & \|u\|_{\tilde{L}_t^\infty(B_{p,r}^s)} \leq C e^{\int_0^t \|\nabla u\|_{L^\infty}} \left( \|u_0\|_{B_{p,r}^s} + (1 + \nu T)^{1-\frac{1}{\rho}} \nu^{\frac{1}{\rho}-1} \|g\|_{\tilde{L}_t^\rho(B_{p,r}^{s-2+\frac{2}{\rho}})} \right) \\ & \leq C e^{\int_0^t \|\nabla u\|_{L^\infty}} \left( \|u_0\|_{B_{p,r}^s} + (1 + \nu t)^{1-\frac{1}{\rho}} \nu^{\frac{1}{\rho}-1} \|\nabla h + \nabla(\ln(1+h))\nabla u\|_{\tilde{L}_t^\rho(B_{p,r}^{s-2+\frac{2}{\rho}})} \right) \\ & \leq C_1 + C_2 t^{\frac{1}{\rho}} \|h\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1+\frac{2}{\rho}})} + C_2 \|\nabla(\ln(1+h))\nabla u\|_{\tilde{L}_t^\rho(B_{p,r}^{s-2+\frac{2}{\rho}})} \\ & \leq C_1 + C_2 t^{\frac{1}{\rho}} \|h\|_{\tilde{L}_t^\infty(B_{p,2}^{s-\varepsilon})} \\ & + C_2 (\|\nabla(\ln(1+h))\|_{\tilde{L}_t^\infty(B_{p,r}^{s-2+\frac{2}{\rho}})} \|\nabla u\|_{L_t^\rho(L^\infty)} + \|\nabla u\|_{\tilde{L}_t^\infty(B_{p,r}^{s-2+\frac{2}{\rho}})} \|\nabla(\ln(1+h))\|_{L_t^\rho(L^\infty)}) \\ & \leq C_3 + C_4 (\|h\|_{\tilde{L}_t^\infty(B_{p,2}^{s-\varepsilon})} + \|u\|_{\tilde{L}_t^\infty(B_{p,2}^{s-\varepsilon})}) \\ & < C(s, p, r, \nu, T^*, \|u_0\|_{B_{p,r}^s}). \end{aligned}$$

For  $h$ , in the case of  $r \geq 2$ , by Lemma 2.13, we have, for any  $t < T^*$

$$\begin{aligned}
(4.9) \quad & \|h\|_{\tilde{L}_t^\infty(B_{p,r}^s)} \leq \exp\left(C \int_0^t \|\nabla u\|_{B_{p,r}^{s-1}} dt'\right) (\|h_0\|_{B_{p,r}^s} + \\
& \|\operatorname{div} u\|_{\tilde{L}_t^1(B_{p,r}^s)} + \|h \operatorname{div} u\|_{\tilde{L}_t^1(B_{p,r}^s)}) \\
& \leq \exp\left(C \int_0^t \|u\|_{B_{p,r}^s} dt'\right) (\|h_0\|_{B_{p,r}^s} + T^{*\frac{1}{2}} \|\operatorname{div} u\|_{\tilde{L}_t^2(B_{p,r}^s)} + \\
& \|h\|_{L_t^2(L^\infty)} \|\operatorname{div} u\|_{\tilde{L}_t^2(B_{p,r}^s)} + \|\operatorname{div} u\|_{L_t^2(L^\infty)} \|h\|_{\tilde{L}_t^2(B_{p,r}^s)}) \\
& \leq C(1 + \|h\|_{L_t^2(B_{p,r}^s)} + \|\operatorname{div} u\|_{\tilde{L}_t^2(B_{p,r}^s)}).
\end{aligned}$$

In view of Lemma 2.14, we obtain that

$$\begin{aligned}
(4.10) \quad & \|\operatorname{div} u\|_{\tilde{L}_t^2(B_{p,r}^s)} \leq C\|u\|_{\tilde{L}_t^2(B_{p,r}^{s+1})} \\
& \leq C\nu^{-\frac{1}{2}} e^{C(1+\nu t)\frac{1}{2} \int_0^t \|\nabla u\|_{L^\infty} dt'} \left( (1+\nu t)^{\frac{1}{2}} \|u_0\|_{B_{p,r}^s} \right. \\
& \left. + (1+\nu t)\nu^{-\frac{1}{2}} \|\nabla h + \nabla(\ln(1+h))\nabla u\|_{\tilde{L}_t^2(B_{p,r}^{s-1})} \right) \\
& \leq C(1 + \|h\|_{\tilde{L}_t^2(B_{p,r}^s)} + \|\nabla(\ln(1+h))\nabla u\|_{\tilde{L}_t^2(B_{p,r}^{s-1})}) \\
& \leq C(1 + \|h\|_{\tilde{L}_t^2(B_{p,r}^s)} + \|\nabla(\ln(1+h))\|_{L_t^2(L^\infty)} \|\nabla u\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \\
& + \|\nabla u\|_{L_t^\infty(L^\infty)} \|\nabla(\ln(1+h))\|_{\tilde{L}_t^2(B_{p,r}^{s-1})}) \\
& \leq C(1 + \|h\|_{\tilde{L}_t^2(B_{p,r}^s)} + \|\nabla(\ln(1+h))\|_{L_t^2(B_{p,r}^{s-1})} \|\nabla u\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \\
& + \|\nabla u\|_{L_t^\infty(B_{p,r}^{s-1})} \|\nabla(\ln(1+h))\|_{\tilde{L}_t^2(B_{p,r}^{s-1})}) \\
& \leq C(1 + \|h\|_{\tilde{L}_t^2(B_{p,r}^s)} + \|\ln(1+h)\|_{L_t^2(B_{p,r}^s)} + \|\ln(1+h)\|_{\tilde{L}_t^2(B_{p,r}^s)}) \\
& \leq C(1 + \|h\|_{L_t^2(B_{p,r}^s)}).
\end{aligned}$$

Combining (4.9), we have

$$(4.11) \quad \|h\|_{\tilde{L}_t^\infty(B_{p,r}^s)} \leq C(1 + \|h\|_{L_t^2(B_{p,r}^s)}).$$

Then it follows

$$\begin{aligned}
(4.12) \quad & \|h\|_{\tilde{L}_t^\infty(B_{p,r}^s)}^2 \leq C + C \int_0^t \|h(t')\|_{B_{p,r}^s}^2 dt' \\
& \leq C + C \int_0^t \|h(t')\|_{\tilde{L}_t^\infty(B_{p,r}^s)}^2 dt',
\end{aligned}$$

where  $C$  in (4.9)-(4.12) only depends on  $s, p, r, \nu, T^*$ ,  $\|h_0\|_{B_{p,r}^s}$ ,  $\|u\|_{\tilde{L}_t^\infty(B_{p,r}^s)}$ ,  $\|h\|_{L_T^2(L^\infty)}$ ,  $\|\operatorname{div} u\|_{L_T^2(L^\infty)}$ .

By the virtue of the Gronwall inequality, we have

$$(4.13) \quad \|h\|_{\tilde{L}_t^\infty(B_{p,r}^s)}^2 \leq C e^{CT}.$$

Reagrds as  $r < 2$ , we rewrite (4.9) as follows

$$\begin{aligned}
(4.14) \quad & \|h\|_{\tilde{L}_t^\infty(B_{p,r}^s)} \leq \exp(C \int_0^t \|\nabla u\|_{B_{p,r}^{s-1}} dt') (\|h_0\|_{B_{p,r}^s} + \\
& \|div u\|_{\tilde{L}_t^1(B_{p,r}^s)} + \|h div u\|_{\tilde{L}_t^1(B_{p,r}^s)}) \\
& \leq \exp(C \int_0^t \|u\|_{B_{p,r}^s} dt') (\|h_0\|_{B_{p,r}^s} + T^{\frac{1}{r-1}} \|div u\|_{\tilde{L}_t^1(B_{p,r}^s)} + \\
& \|h\|_{L_t^{r'}(L^\infty)} \|div u\|_{\tilde{L}_t^1(B_{p,r}^s)} + \|div u\|_{L_t^{r'}(L^\infty)} \|h\|_{\tilde{L}_t^1(B_{p,r}^s)}) \\
& \leq C(1 + \|h\|_{L_t^r(B_{p,r}^s)} + \|div u\|_{\tilde{L}_t^1(B_{p,r}^s)}).
\end{aligned}$$

In view of Lemma 2.14, we obtain that

$$\begin{aligned}
(4.15) \quad & \|div u\|_{\tilde{L}_t^r(B_{p,r}^s)} \leq C \|u\|_{\tilde{L}_t^r(B_{p,r}^{s+1})} \\
& \leq C \nu^{-\frac{1}{r}} e^{C(1+\nu t) \int_0^t \|\nabla u\|_{L^\infty} dt'} \left( (1 + \nu t)^{\frac{1}{r}} \|u_0\|_{B_{p,r}^{s+1-\frac{2}{r}}} \right. \\
& \left. + (1 + \nu t) \nu^{\frac{1}{r}-1} \|\nabla h + \nabla(\ln(1+h))\nabla u\|_{\tilde{L}_t^r(B_{p,r}^{s-1})} \right) \\
& \leq C(1 + \|h\|_{\tilde{L}_t^r(B_{p,r}^s)} + \|\nabla(\ln(1+h))\nabla u\|_{\tilde{L}_t^r(B_{p,r}^{s-1})}) \\
& \leq C(1 + \|h\|_{\tilde{L}_t^r(B_{p,r}^s)} + \|\nabla(\ln(1+h))\|_{L_t^r(L^\infty)} \|\nabla u\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \\
& + \|\nabla u\|_{L_t^\infty(L^\infty)} \|\nabla(\ln(1+h))\|_{\tilde{L}_t^r(B_{p,r}^{s-1})}) \\
& \leq C(1 + \|h\|_{\tilde{L}_t^r(B_{p,r}^s)} + \|\nabla(\ln(1+h))\|_{L_t^r(B_{p,r}^{s-1})} \|\nabla u\|_{\tilde{L}_t^\infty(B_{p,r}^{s-1})} \\
& + \|\nabla u\|_{L_t^\infty(B_{p,r}^{s-1})} \|\nabla(\ln(1+h))\|_{\tilde{L}_t^r(B_{p,r}^{s-1})}) \\
& \leq C(1 + \|h\|_{\tilde{L}_t^r(B_{p,r}^s)} + \|\ln(1+h)\|_{L_t^r(B_{p,r}^s)} + \|\ln(1+h)\|_{\tilde{L}_t^r(B_{p,r}^s)}) \\
& \leq C(1 + \|h\|_{L_t^r(B_{p,r}^s)}).
\end{aligned}$$

Combining (4.14), we have

$$(4.16) \quad \|h\|_{\tilde{L}_t^\infty(B_{p,r}^s)} \leq C(1 + \|h\|_{L_t^r(B_{p,r}^s)}).$$

Then it follows

$$\begin{aligned}
(4.17) \quad & \|h\|_{\tilde{L}_t^\infty(B_{p,r}^s)}^r \leq C + C \int_0^t \|h(t')\|_{B_{p,r}^s}^r dt' \\
& \leq C + C \int_0^t \|h(t')\|_{\tilde{L}_t^\infty(B_{p,r}^s)}^r dt',
\end{aligned}$$

where  $C$  in (4.14)-(4.17) only depends on  $s, p, r, \nu, T^*$ ,  $\|h_0\|_{B_{p,r}^s}$ ,  $\|u\|_{\tilde{L}_t^\infty(B_{p,r}^s)}$ ,  $\|h\|_{L_T^2(L^\infty)}$ ,  $\|div u\|_{L_T^2(L^\infty)}$ .

By the viture of the Gronwall inequality, we have

$$(4.18) \quad \|h\|_{\tilde{L}_t^\infty(B_{p,r}^s)}^r \leq C e^{CT}.$$

Combining Theorem 1.1 and the continuity of  $u, h$  in  $B_{p,r}^s$  completes the proof of Proposition 4.2.

#### 4.3. Global existence

At last, we give some corollaries about the global existence.

**Corollary 4.3.** *Let  $u_0, h_0 \in B_{p_1, r_1}^{s_1} \cap B_{p_2, r_2}^{s_2}$ ,  $(s_1, p_1, r_1)$  and  $(s_2, p_2, r_2)$  satisfy the same conditions in Theorem 1.2, and let  $T_1, T_2$  be the maximal existence time of the Cauchy problem (2.1) in  $B_{p_1, r_1}^{s_1}$  and  $B_{p_2, r_2}^{s_2}$  respectively, then we have  $T_1 = T_2$ .*

Proof: Assume that  $T_1(< \infty) < T_2$ . By Proposition 4.2, we have

$$\int_0^{T_1} \|\nabla u\|_{L^\infty}^{r_1} + \|h\|_{L^\infty}^{r_1} + \|\nabla(\ln(1+h))\|_{L^\infty}^{r_1} dt' = \infty.$$

On the other hand, by  $T_1 < T_2$ , we have

$$\|u\|_{\tilde{L}_{T_1}^\infty(B_{p_2, r_2}^{s_2})} + \|h\|_{\tilde{L}_{T_1}^\infty(B_{p_2, r_2}^{s_2})} < \infty.$$

In view of  $s_2 > 1 + \frac{2}{p_2}$ , we get

$$\|u(t)\|_{L^\infty} + \|h(t)\|_{L^\infty} + \|\nabla u(t)\|_{L^\infty} + \|\nabla \ln(1+h(t))\|_{L^\infty} \leq C(\|u(t)\|_{B_{p_2, r_2}^{s_2}} + \|u(t)\|_{B_{p_2, r_2}^{s_2}}).$$

Then we obtain

$$\begin{aligned} & \int_0^{T_1} \|\nabla u\|_{L^\infty}^{r_1} + \|h\|_{L^\infty}^{r_1} + \|\nabla(\ln(1+h))\|_{L^\infty}^{r_1} dt' \\ (4.19) \quad & \leq C \int_0^{T_1} (\|u\|_{\tilde{L}_{T_1}^\infty(B_{p_2, r_2}^{s_2})} + \|h\|_{\tilde{L}_{T_1}^\infty(B_{p_2, r_2}^{s_2})}) dt' \\ & < \infty, \end{aligned}$$

which leads to a contradiction. So we have  $T_1 \geq T_2$ . Of course, we also have  $T_2 \geq T_1$  by the same argument.

**Corollary 4.4.** *Let  $u_0, h_0 \in B_{p,r}^s \cap H^{s_1}$ ,  $(s, p, r)$  satisfy the same condition in Corollary 1.2,  $s_1 > 2$ . If there exists an  $\varepsilon$  small enough, such that  $\|u_0\|_{H^{s_1}} + \|h_0\|_{H^{s_1}} < \varepsilon$ . Then the corresponding solution of the Cauchy problem (2.1) in  $B_{p,r}^s$  is global in time.*

Proof: It's an obvious conclusion of Corollary 4.3 and Lemma 2.18.

**Proof of Theorem 1.3:** By  $s > 1 + \frac{2}{p}$ , there exists a real number  $s'$  such that  $s > s' > 2$ . Then the space  $B_{p,r}^s$  is continuous embedding in  $H^{s'}$ , and for any  $u, h \in B_{p,r}^s$ , we have

$$\|u\|_{H^{s'}} + \|h\|_{H^{s'}} \leq C(\|u\|_{B_{p,r}^s} + \|h\|_{B_{p,r}^s}).$$

Thus by Corollary 4.3 and Lemma 2.18, we obtain the desired result in Theorem 1.3. This completes the proof.

**Remark 4.1** Note that Wang and Xu in [12] established the local well-posedness of the system (1.1) and got the global solutions to the system (1.1) for small initial data in Sobolev spaces  $H^s$ ,  $s > 2$ . Our obtained results in Theorem 1.1 with  $p = 2$  and  $r = 2$  cover the recent results in [12].

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